Topological Free Entropy Dimension in Unital C*-algebras

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Abstract: The notion of topological free entropy dimension of n-tuple of elements in a unital C* algebra was introduced by Voiculescu. In the paper, we compute topological free entropy dimension of one self-adjoint element and topological free orbit dimension of one self-adjoint element in a unital C* algebra. We also calculate the values of topological free entropy dimensions of any families of self-adjoint generators of some unital C* algebras, including irrational rotation C* algebra, UHF algebra, and minimal tensor product of two reduced C* algebras of free groups.

Keywords: Topological free entropy dimension, C* algebra

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1. Introduction

The theory of free probability and free entropy was developed by Voiculescu from 1990s. It played a crucial role in the recent study of finite von Neumann algebras (see [**1**], [**3**], [**4**], [**5**], [**6**], [**7**], [**8**], [**11**], [**14**], [**15**], [**23**], [**24**], [**25**]). The analogue of free entropy dimension in C* algebra context, the notion of topological free entropy dimension of of $n-$tuple of elements in a unital C* algebra, was also introduced by Voiculescu in [**26**].

After introducing the concept of topological free entropy dimension of n-tuple of elements in a unital C* algebra, Voiculescu discussed some of its properties including subadditivity and change of variables in [**26**]. In this paper, we will add one basic property into the list: topological free entropy dimension of one variable. More specifically, suppose x is a self-adjoint element in a unital C* algebra A and $\sigma(x)$ is the spectrum of x in $\mathcal{A}$. Then topological free entropy dimension of x is equal to $1 - \frac{1}{n}$ where n is the cardinality of the set $\sigma(x)$ (see Theorem 4.1).

In [**26**], Voiculescu showed that (i) if $x_1, \ldots, x_n$ is a family of free semicircular elements in a unital C* algebra with a tracial state, then $\delta_{top}(x_1, \ldots, x_n) = n$, where $\delta_{top}(x_1, \ldots, x_n)$ is the topological free entropy dimension of $x_1, \ldots, x_n$; (ii) if $x_1, \ldots, x_n$ is the universal n-tuple of self-adjoint contractions, then $\delta_{top}(x_1, \ldots, x_n) = n$. Except in these two cases, very few has been known on the values of topological free entropy dimensions in other C* algebras. Using the inequality between topological free entropy dimension and Voiculescu's free dimension capacity, we are able to obtain an estimation of upper-bound of topological free entropy dimension for a

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unital C* algebra with a unique tracial state (see Theorem 5.1). The lower-bound of topological free entropy dimension is also obtained for infinite dimensional simple unital C* algebra with a unique tracial state (see Theorem 5.2). As a corollary, we know that the topological free entropy dimension of any family of self-adjoint generators of an irrational rotation C* algebra or UHF algebra or $C^*_{red}(F_2)\otimes_{min} C^*_{red}(F_2)$ is equal to 1 (see Theorem 5.3, 5.4, 5.5). For these C* algebras, the value of the topological free entropy dimension is independent of the choice of generators.

The rest of the paper is devoted to study another invariant associated to n-tuple of elements in C* algebras. This invariant, called topological free orbit dimension, is an analogue of free orbit dimension in finite von Neumann algebras (see [**11**]). We show that the topological free orbit dimension of a self-adjoint element in a unital C* algebra is equal to, according to some measurement, the packing dimension of the spectrum of x (see Theorem 7.1).

The organization of the paper is as follows. In the section 2, we recall the definition of topological free entropy dimension. Some technical lemmas are proved in section 3. In section 4, we compute the topological free entropy dimension of one self-adjoint element in a unital C* algebra. In section 5, we study the relationship between topological free entropy dimension and free capacity dimension of a unital C* algebra. Then we show that topological free entropy dimension of of any family of generators of an infinite dimensional simple unital C* algebra with a unique tracial state is always greater than or equal to 1. The concept of topological free orbit dimension of n-tuple of elements in a C* algebra is introduced in section 6. Its value for one variable is computed in section 7.

2. Definitions and preliminary

In this section, we are going to recall Voiculescu's definition of topological free entropy dimension of n-tuple of elements in a unital C* algebra.

2.1. A Covering of a set in a metric space. Suppose (X, d) is a metric space and K is a subset of X. A family of balls in X is called a covering of K if the union of these balls covers K and the centers of these balls lie in K.

2.2. Covering numbers in complex matrix algebra $(\mathcal{M}_k(\mathbb{C}))^n$. Let $\mathcal{M}_k(\mathbb{C})$ be the $k \times k$ full matrix algebra with entries in $\mathbb{C}$, and τ_k be the normalized trace on $\mathcal{M}_k(\mathbb{C})$, i.e., $\tau_k = \frac{1}{k}Tr$, where Tr is the usual trace on $\mathcal{M}_k(\mathbb{C})$. Let $\mathcal{U}(k)$ denote the group of all unitary matrices in $\mathcal{M}_k(\mathbb{C})$. Let $\mathcal{M}_k(\mathbb{C})^n$ denote the direct sum of n copies of $\mathcal{M}_k(\mathbb{C})$. Let $\mathcal{M}_k^{s.a}(\mathbb{C})$ be the subalgebra of $\mathcal{M}_k(\mathbb{C})$ consisting of all self-adjoint matrices of $\mathcal{M}_k(\mathbb{C})$. Let $(\mathcal{M}_k^{s.a}(\mathbb{C}))^n$ be the direct sum of n copies of $\mathcal{M}_k^{s.a}(\mathbb{C})$. Let $\|\cdot\|$ be an operator norm on $\mathcal{M}_k(\mathbb{C})^n$ defined by

$$\|(A_1, \ldots, A_n)\| = \max\{\|A_1\|, \ldots, \|A_n\|\}$$

for all $(A_1, \ldots, A_n)$ in $\mathcal{M}_k(\mathbb{C})^n$. Let $\|\cdot\|_2$ denote the trace norm induced by τ_k on $\mathcal{M}_k(\mathbb{C})^n$, i.e.,

$$\|(A_1, \ldots, A_n)\|_2 = \sqrt{\tau_k(A_1^*A_1) + \ldots + \tau_k(A_n^*A_n)}$$

for all $(A_1, \ldots, A_n)$ in $\mathcal{M}_k(\mathbb{C})^n$.

For every $\omega > 0$, we define the ω-$\|\cdot\|$-ball $Ball(B_1, \ldots, B_n; \omega, \|\cdot\|)$ centered at $(B_1, \ldots, B_n)$ in $\mathcal{M}_k(\mathbb{C})^n$ to be the subset of $\mathcal{M}_k(\mathbb{C})^n$ consisting of all $(A_1, \ldots, A_n)$ in $\mathcal{M}_k(\mathbb{C})^n$ such that

$$\|(A_1, \ldots, A_n) - (B_1, \ldots, B_n)\| < \omega.$$

DEFINITION 2.1. *Suppose that Σ is a subset of $\mathcal{M}_k(\mathbb{C})^n$. We define the covering number $\nu_\infty(\Sigma, \omega)$ to be the minimal number of ω-$\|\cdot\|$-balls that consist a covering of Σ in $\mathcal{M}_k(\mathbb{C})^n$.*

For every $\omega > 0$, we define the ω-$\|\cdot\|_2$-ball $Ball(B_1, \ldots, B_n; \omega, \|\cdot\|_2)$ centered at $(B_1, \ldots, B_n)$ in $\mathcal{M}_k(\mathbb{C})^n$ to be the subset of $\mathcal{M}_k(\mathbb{C})^n$ consisting of all $(A_1, \ldots, A_n)$ in $\mathcal{M}_k(\mathbb{C})^n$ such that

$$\|(A_1, \ldots, A_n) - (B_1, \ldots, B_n)\|_2 < \omega.$$

DEFINITION 2.2. *Suppose that Σ is a subset of $\mathcal{M}_k(\mathbb{C})^n$. We define the covering number $\nu_2(\Sigma, \omega)$ to be the minimal number of ω-$\|\cdot\|_2$-balls that consist a covering of Σ in $\mathcal{M}_k(\mathbb{C})^n$.*

2.3. Noncommutative polynomials. In this article, we always assume that $\mathcal{A}$ is a unital C*-algebra. Let $x_1, \ldots, x_n, y_1, \ldots, y_m$ be self-adjoint elements in $\mathcal{A}$. Let $\mathbb{C}\langle X_1, \ldots, X_n, Y_1, \ldots, Y_m\rangle$ be the unital noncommutative polynomials in the indeterminates $X_1, \ldots, X_n, Y_1, \ldots, Y_m$. Let $\{P_r\}_{r=1}^\infty$ be the collection of all noncommutative polynomials in $\mathbb{C}\langle X_1, \ldots, X_n, Y_1, \ldots, Y_m\rangle$ with rational complex coefficients. (Here "rational complex coefficients" means that the real and imaginary parts of all coefficients of P_r are rational numbers).

REMARK 2.1. *We alsways assume that $1 \in \mathbb{C}\langle X_1, \ldots, X_n, Y_1, \ldots, Y_m\rangle$.*

2.4. Voiculescu's Norm-microstates Space. For all integers $r, k \geq 1$, real numbers $R, \epsilon > 0$ and noncommutative polynomials $P_1, \ldots, P_r$, we define

$$\Gamma_R^{(top)}(x_1, \ldots, x_n, y_1, \ldots, y_m; k, \epsilon, P_1, \ldots, P_r)$$

to be the subset of $(\mathcal{M}_k^{s.a}(\mathbb{C}))^{n+m}$ consisting of all these

$$(A_1, \ldots, A_n, B_1, \ldots, B_m) \in (\mathcal{M}_k^{s.a}(\mathbb{C}))^{n+m}$$

satisfying

$$\max\{\|A_1\|, \ldots, \|A_n\|, \|B_1\|, \ldots, \|B_m\|\} \leq R$$

and

$$|\|P_j(A_1, \ldots, A_n, B_1, \ldots, B_m)\| - \|P_j(x_1, \ldots, x_n, y_1, \ldots, y_m)\|| \leq \epsilon, \qquad \forall\ 1 \leq j \leq r.$$

REMARK 2.2. *In the definition of norm-microstates space, we use the following assumption. If*

$$P_j(x_1, \ldots, x_n, y_1, \ldots, y_m) = \alpha_0 \cdot I_{\mathcal{A}} + \sum_{s=1}^{N} \sum_{1 \leq i_1, \ldots, i_s \leq n+m} \alpha_{i_1 \cdots i_s} z_{i_1} \cdots z_{i_s}$$

where $z_1, \ldots, z_{n+m}$ denotes $x_1, \ldots, x_n, y_1, \ldots, y_m$ and $\alpha_0, \alpha_{i_1 \cdots i_s}$ are in $\mathbb{C}$, then

$$P_j(A_1, \ldots, A_n, B_1, \ldots, B_m) = \alpha_0 \cdot I_k + \sum_{s=1}^{N} \sum_{1 \leq i_1, \ldots, i_s \leq n+m} \alpha_{i_1 \cdots i_s} Z_{i_1} \cdots Z_{i_s}$$

where $Z_1, \ldots, Z_{n+m}$ denotes $A_1, \ldots, A_n, B_1, \ldots, B_m$ and I_k is the identity matrix in $\mathcal{M}_k(\mathbb{C})$.

Remark 2.3. *In the original definition of norm-microstates space in* [**26**]*, the parameter R was not introduced. Note the following observation: Let $R > \max\{\|x_1\|, \ldots, \|x_n\|, \|y_1\|, \ldots, \|y_m\|\}$. When r is large enough so that*

$$\{X_1, \ldots, X_n, Y_1, \ldots, Y_m\} \subset \{P_1, \ldots, P_r\}$$

and $0 < \epsilon < R - \max\{\|x_1\|, \ldots, \|x_n\|, \|y_1\|, \ldots, \|y_m\|\}$, we have

$$\Gamma_R^{(top)}(x_1, \ldots, x_n, y_1, \ldots, y_m; k, \epsilon, P_1, \ldots, P_r) = \Gamma_{top}(x_1, \ldots, x_n, y_1, \ldots, y_m; k, \epsilon, P_1, \ldots, P_r)$$

for all $k \geq 1$, where $\Gamma_{(top)}(x_1, \ldots, x_n, y_1, \ldots, y_m; k, \epsilon, P_1, \ldots, P_r)$ is the norm-microstates space defined in [**26**]*. Thus our definition agrees with the one in* [**26**] *for large R, r and small ϵ.*

In the later sections, we need to construct the ultraproduct of some matrix algebras, it will be convenient for us to include the parameter "R" in the definition of norm-microstate space.

Define the norm-microstates space of $x_1, \ldots, x_n$ in the presence of $y_1, \ldots, y_m$, denoted by

$$\Gamma_R^{(top)}(x_1, \ldots, x_n : y_1, \ldots, y_m; k, \epsilon, P_1, \ldots, P_r)$$

as the projection of $\Gamma_R^{(top)}(x_1, \ldots, x_n, y_1, \ldots, y_m; k, \epsilon, P_1, \ldots, P_r)$ onto the space $(\mathcal{M}_k^{s.a}(\mathbb{C}))^n$ via the mapping

$$(A_1, \ldots, A_n, B_1, \ldots, B_m) \to (A_1, \ldots, A_n).$$

2.5. Voiculescu's topological free entropy dimension (see [26]). Define

$$\nu_\infty(\Gamma_R^{(top)}(x_1, \ldots, x_n : y_1, \ldots, y_m; k, \epsilon, P_1, \ldots, P_r), \omega)$$

to be the covering number of the set $\Gamma_R^{(top)}(x_1, \ldots, x_n : y_1, \ldots, y_m; k, \epsilon, P_1, \ldots, P_r)$ by ω-$\|\cdot\|$-balls in the metric space $(\mathcal{M}_k^{s.a}(\mathbb{C}))^n$ equipped with operator norm.

Definition 2.3. *Define*

$$\delta_{top}(x_1, \ldots, x_n : y_1, \ldots, y_m; \omega) = \sup_{R>0} \inf_{\epsilon>0, r\in\mathbb{N}} \limsup_{k\to\infty} \frac{\log(\nu_\infty(\Gamma_R^{(top)}(x_1, \ldots, x_n : y_1, \ldots, y_m; k, \epsilon, P_1, \ldots, P_r), \omega))}{-k^2 \log \omega}.$$

The topological entropy dimension *of $x_1, \ldots, x_n$ in the presence of $y_1, \ldots, y_m$ is defined by*

$$\delta_{top}(x_1, \ldots, x_n : y_1, \ldots, y_m) = \limsup_{\omega\to 0^+} \delta_{top}(x_1, \ldots, x_n : y_1, \ldots, y_m; \omega).$$

Remark 2.4. *Let $M > \max\{\|x_1\|, \ldots, \|x_n\|, \|y_1\|, \ldots, \|y_m\|\}$ be some positive number. By Remark 2.3, we know*

$$\delta_{top}(x_1, \ldots, x_n : y_1, \ldots, y_m) = \limsup_{\omega\to 0^+} \inf_{\epsilon>0, r\in\mathbb{N}} \limsup_{k\to\infty} \frac{\log(\nu_\infty(\Gamma_M^{(top)}(x_1, \ldots, x_n : y_1, \ldots, y_m; k, \epsilon, P_1, \ldots, P_r), \omega))}{-k^2 \log \omega}.$$

2.6. C* algebra ultraproduct and von Neumann algebra ultraproduct. Suppose $\{\mathcal{M}_{k_m}(\mathbb{C})\}_{m=1}^{\infty}$ is a sequence of complex matrix algebras where k_m goes to infinity when m approaches infinity. Let γ be a free ultrafilter in $\beta(\mathbb{N})\setminus\mathbb{N}$. We can introduce a unital C* algebra $\prod_{m=1}^{\infty}\mathcal{M}_{k_m}(\mathbb{C})$ as follows:

$$\prod_{m=1}^{\infty}\mathcal{M}_{k_m}(\mathbb{C}) = \{(Y_m)_{m=1}^{\infty} \mid \forall\, m\geq 1,\ Y_m\in\mathcal{M}_{k_m}(\mathbb{C}) \ \text{ and } \ \sup_{m\geq 1}\|Y_m\|<\infty\}.$$

We can also introduce the norm closed two sided ideals $\mathcal{I}_\infty$ and $\mathcal{I}_2$ as follows.

$$\mathcal{I}_\infty = \{(Y_m)_{m=1}^{\infty}\in\prod_{m=1}^{\infty}\mathcal{M}_{k_m}(\mathbb{C}) \mid \lim_{m\to\gamma}\|Y_m\|=0\}$$
$$\mathcal{I}_2 = \{(Y_m)_{m=1}^{\infty}\in\prod_{m=1}^{\infty}\mathcal{M}_{k_m}(\mathbb{C}) \mid \lim_{m\to\gamma}\|Y_m\|_2=0\}$$

Definition 2.4. *The C* algebra ultraproduct of $\{\mathcal{M}_{k_m}(\mathbb{C})\}_{m=1}^{\infty}$ along the ultrfilter γ, denoted by $\prod_{m=1}^{\gamma}\mathcal{M}_{k_m}(\mathbb{C})$, is defined to be the quotient algebra of $\prod_{m=1}^{\infty}\mathcal{M}_{k_m}(\mathbb{C})$ by the ideal $\mathcal{I}_\infty$. The image of $(Y_m)_{m=1}^{\infty}\in\prod_{m=1}^{\infty}\mathcal{M}_{k_m}(\mathbb{C})$ in the quotient algebra is denoted by $[(Y_m)_m]$.*

Definition 2.5. *The von Neumann algebra ultraproduct of $\{\mathcal{M}_{k_m}(\mathbb{C})\}_{m=1}^{\infty}$ along the ultrfilter γ, also denoted by $\prod_{m=1}^{\gamma}\mathcal{M}_{k_m}(\mathbb{C})$ if no confusion arises, is defined to be the quotient algebra of $\prod_{m=1}^{\infty}\mathcal{M}_{k_m}(\mathbb{C})$ by the ideal $\mathcal{I}_2$. The image of $(Y_m)_{m=1}^{\infty}\in\prod_{m=1}^{\infty}\mathcal{M}_{k_m}(\mathbb{C})$ in the quotient algebra is denoted by $[(Y_m)_m]$.*

Remark 2.5. *The von Neumann algebra ultraproduct $\prod_{m=1}^{\gamma}\mathcal{M}_{k_m}(\mathbb{C})$ is a finite factor (see* [**16**]*).*

2.7. Topological free entropy dimension of elements in a non-unital C* algebra. Topological free entropy dimension can also be defined for n-tuple of elements in a non-unital C* algebra. Suppose that $\mathcal{A}$ is a non-unital C*-algebra. Let $x_1,\ldots,x_n,y_1,\ldots,y_m$ be self-adjoint elements in $\mathcal{A}$. Let $\mathbb{C}\langle X_1,\ldots,X_n,Y_1,\ldots,Y_m\rangle\ominus\mathbb{C}$ be the noncommutative polynomials in the indeterminates $X_1,\ldots,X_n,Y_1,\ldots,Y_m$ without constant terms. Let $\{P_r\}_{r=1}^{\infty}$ be the collection of all noncommutative polynomials in $\mathbb{C}\langle X_1,\ldots,X_n,Y_1,\ldots,Y_m\rangle\ominus\mathbb{C}$ with rational complex coefficients. Then norm-mocrostate space

$$\Gamma_R^{(top)}(x_1,\ldots,x_n:y_1,\ldots,y_m;k,\epsilon,P_1,\ldots,P_r)$$

can be defined similarly as in section 2.4. So topological free entropy dimension

$$\delta_{top}(x_1,\ldots,x_n:y_1,\ldots,y_m)$$

can also be defined similarly as in section 2.5.

In the paper, we will focus on the case when $\mathcal{A}$ is a unital C* algebra.

3. Some technical lemmas

3.1. Suppose x is a self-adjoint element in a unital C* algebra $\mathcal{A}$. Let $\sigma(x)$ be the spectrum of x in $\mathcal{A}$.

THEOREM 3.1. *Let $R > \|x\|$. For any $\omega > 0$, we have the following.*

(1) *There are some integer $n \geq 1$ and distinct real numbers $\lambda_1, \lambda_2, \cdots, \lambda_n$ in $\sigma(x)$ satisfying (i) $|\lambda_i - \lambda_j| \geq \omega$ for all $1 \leq i \neq j \leq n$; and (ii) for any λ in $\sigma(x)$, there is some λ_j with $1 \leq j \leq n$ such that $|\lambda - \lambda_j| \leq \omega$.*

(2) *There are some $r_0 > 0$ and $\epsilon_0 > 0$ such that the following holds: when $r > r_0$, $\epsilon < \epsilon_0$, for any A in $\Gamma_R^{(top)}(x; k, \epsilon, P_1, \ldots, P_r)$, there are positive integers $1 \leq k_1, \ldots, k_n \leq k$ with $k_1 + k_2 + \cdots + k_n = k$ and some unitary matrix U in $\mathcal{M}_k(\mathbb{C})$ satisfying*

$$\|U^*AU - \begin{pmatrix} \lambda_1 I_{k_1} & 0 & \cdots & 0 \\ 0 & \lambda_2 I_{k_2} & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & \cdots & \lambda_n I_{k_n} \end{pmatrix}\| \leq 2\omega,$$

where I_{k_j} is the $k_j \times k_j$ identity matrix in $\mathcal{M}_{k_j}(\mathbb{C})$ for $1 \leq j \leq n$.

PROOF. The proof of part (1) is trivial. We will only prove part (2). Assume that the result in (2) does not hold. Then there is some $\omega > 0$ so that the following holds: for all $m \geq 1$, there are $k_m \geq 1$ and some A_m in $\Gamma_R^{(top)}(x; k_m, \frac{1}{m}, P_1, \ldots, P_m)$ such that

$$\|U^*A_mU - \begin{pmatrix} \lambda_1 I_{s_1} & 0 & \cdots & 0 \\ 0 & \lambda_2 I_{s_2} & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & \cdots & \lambda_n I_{s_n} \end{pmatrix}\| > 2\omega, \qquad (*)$$

for every $1 \leq s_1, \ldots, s_n \leq k_m$ with $s_1 + \cdots + s_n = k_m$ and every unitary matrix U in $\mathcal{M}_{k_m}(\mathbb{C})$.

Let γ be a free ultrafilter in $\beta(\mathbb{N})\setminus\mathbb{N}$. Let $\mathcal{B} = \prod_{m=1}^{\gamma} \mathcal{M}_{k_m}(\mathbb{C})$ be the C* algebra ultraproduct of $\{\mathcal{M}_{k_m}(\mathbb{C})\}_{m=1}^{\infty}$ along the ultrafilter α, i.e. $\prod_{m=1}^{\gamma} \mathcal{M}_{k_m}(\mathbb{C})$ is the quotient algebra of the C* algebra $\prod_{m=1}^{\infty} \mathcal{M}_{k_m}(\mathbb{C})$ by $\mathcal{I}_\infty$, the 0-ideal of the norm $\|\cdot\|$, where $\mathcal{I}_\infty = \{(A_m)_{m=1}^{\infty} \in \prod_{m=1}^{\infty} \mathcal{M}_{k_m}(\mathbb{C}) \mid \lim_{m\to\gamma} \|A_m\| = 0\}$. Let $a = [(U^*A_mU)_{m=1}^{\infty}]$ be a self-adjoint element in $\mathcal{B}$. By mapping x to a, there is a unital $*$-isomorphism from the C* subalgebra generated by $\{I_{\mathcal{A}}, x\}$ in $\mathcal{A}$ onto the C* subalgebra generated by $\{I_{\mathcal{B}}, a\}$ in $\mathcal{B}$. Thus $\sigma(x) = \sigma(a)$. It is not hard to see that Hausdorff-dist$(\sigma(U^*A_mU), \sigma(a)) \to 0$ as m goes to γ, which contradicts with the results in part (1) and $(*)$.

□

3.2. In this subsection, we will use the following notation.

(i) Let n, m be some positive integers with $n \geq m$.

(ii) Let δ, θ be some positive numbers.

(iii) Let $\{\lambda_1, \lambda_2, \ldots, \lambda_m\} \cup \{\lambda_{m+1}, \ldots, \lambda_n\}$ be a family of real numbers such that

$$|\lambda_i - \lambda_j| \geq \theta \qquad \text{for all } 1 \leq i < j \leq m.$$

(iv) Let k be a positive integer such that $k - (n-m)$ is divided by m. We let

$$t = \frac{k-n+m}{m}.$$

(v) We let

$$B = diag(\lambda_{m+1}, \ldots, \lambda_n)$$

be a diagonal matrix in $\mathcal{M}_{n-m}(\mathbb{C})$ and

$$A = diag(\lambda_1 I_t, \lambda_2 I_t, \ldots, \lambda_m I_t, B)$$

be a block-diagonal matrix in $\mathcal{M}_k(\mathbb{C})$, where I_t is the identity matrix in $\mathcal{M}_t(\mathbb{C})$.

(vi) We let A be defined as above and

$$\Omega(A) = \{U^*AU \mid U \text{ is in } \mathcal{U}(k)\}.$$

(vii) Assume that $\{e_{ij}\}_{i,j=1}^k$ is a canonical basis of $\mathcal{M}_k(\mathbb{C})$. We let

$$V_1 = span\{e_{ij} \mid |\lambda_{[\frac{i}{m}]+1} - \lambda_{[\frac{j}{m}]+1}| \geq \theta, \text{ with } 1 \leq i,j < mt\}; \quad \text{and}$$
$$V_2 = \mathcal{M}_k(\mathbb{C}) \ominus V_1,$$

where $[\frac{i}{m}]$, or $[\frac{j}{m}]$, denotes the largest integer $\leq [\frac{i}{m}]$, or $[\frac{j}{m}]$ respectively.

Lemma 3.1. *We follow the notation as above. Suppose* $\|U_1AU_1^* - U_2AU_2^*\|_2 \leq \delta$ *for some unitary matrices* U_1 *and* U_2 *in* $\mathcal{U}(k)$. *Then the following hold.*

(1) *There exists some* $S \in V_2$ *such that* $\|S\|_2 \leq 1$ *and*

$$\|U_1 - U_2S\|_2 \leq \frac{\delta}{\theta}.$$

(2) *If* $n = m$, *then there is a unitary matrix* W *in* V_2 *such that*

$$\|U_1 - U_2W\|_2 \leq \frac{3\delta}{\theta}.$$

Proof. Assume that

$$U_2^*U_1 = \begin{pmatrix} U_{11} & U_{12} & \cdots & U_{1,m+1} \\ U_{21} & U_{22} & \cdots & U_{2,m+1} \\ \cdots & \cdots & \cdots & \cdots \\ U_{m+1,1} & U_{m+1,2} & \cdots & U_{m+1,m+1} \end{pmatrix}$$

where $U_{i,j}$ is a $t \times t$ matrix, $U_{i,m+1}$ a $t \times (n-m)$ matrix, $U_{m+1,j}$ a $(n-m) \times t$ matrix for $1 \leq i,j \leq m$ and $U_{m+1,m+1}$ is a $(n-m) \times (n-m)$ matrix.

(1) Let

$$S=\begin{pmatrix} U_{11} & 0 & \cdots & 0 & U_{1,m+1} \\ 0 & U_{22} & \cdots & 0 & U_{2,m+1} \\ \cdots & \cdots & \ddots & \cdots & \cdots \\ 0 & 0 & \cdots & U_{m,m} & U_{m,m+1} \\ U_{m+1,1} & U_{m+1,2} & \cdots & U_{m+1,m} & U_{m+1,m+1} \end{pmatrix}.$$

It is easy to see that S is in V_2, $\|S\|_2 \le 1$ and

$$\begin{aligned}\delta^2 &\ge \|U_1AU_1^* - U_2AU_2^*\|_2^2 = \frac{1}{k}Tr((U_2^*U_1A - AU_2^*U_1)(U_2^*U_1A - AU_2^*U_1)^*) \\ &\ge \frac{1}{k}\sum_{1\le i\ne j\le m} Tr(|\lambda_i-\lambda_j|^2 U_{ij}U_{ij}^*) \\ &\ge \frac{1}{k}\cdot\theta^2 \sum_{1\le i\ne j\le m} Tr(U_{ij}U_{ij}^*).\end{aligned}$$

Hence

$$\|U_1 - U_2S\|_2^2 = \|U_2^*U_1 - S\|_2^2 = \frac{1}{k}\sum_{1\le i\ne j\le m} Tr(U_{ij}U_{ij}^*) \le \frac{\delta^2}{\theta^2}.$$

It follows that

$$\|U_1 - U_2S\|_2 \le \frac{\delta}{\theta}.$$

(2) If $n = m$, then

$$V_2 = \mathcal{M}_t(\mathbb{C}) \oplus \mathcal{M}_t(\mathbb{C}) \oplus \cdots \oplus \mathcal{M}_t(\mathbb{C}).$$

By the construction of S, we can assume $S = WH$ is a polar decomposition of S in V_2 for some unitary matrix W and positive matrix H in V_2. Again by the construction of S, we know that $\|S\| \le 1$, whence $\|H\| \le 1$. From the proven fact that $\|U_2^*U_1 - S\|_2 \le \frac{\delta}{\theta}$, we know that

$$\|H^2 - I\|_2 = \|S^*S - I\|_2 \le \frac{2\delta}{\theta}.$$

Thus

$$\|H - I\|_2 \le \|H^2 - I\|_2 \le \frac{2\delta}{\theta}.$$

It follows that

$$\|U_1 - U_2W\|_2 \le \|U_1 - U_2WH\|_2 + \|U_2WH - U_2W\|_2 = \|U_1 - U_2S\|_2 + \|H - I\|_2 \le \frac{3\delta}{\theta}.$$

□

Lemma 3.2. *We have the following results.*

(1) *For every $U \in \mathcal{U}(k)$, let*

$$\Sigma(U) = \{W \in \mathcal{U}(k) \mid \exists\ S \in V_2 \text{ such that } \|S\|_2 \le 1 \quad \text{and} \quad \|W - US\|_2 \le \frac{\delta}{\theta}\}.$$

Then the volume of $\Sigma(U)$ is bounded above by

$$\mu(\Sigma(U)) \le (C_1 \cdot 4\delta/\theta)^{k^2} \cdot \left(\frac{C\theta}{\delta}\right)^{2mt^2+4m(n-m)t+2(n-m)^2},$$

where μ is the normalized Haar measure on the unitary group $\mathcal{U}(k)$ and C, C_1 are some constants independent of k, δ, θ.

(2) *When $n = m$, for every $U \in \mathcal{U}(k)$, let*

$$\tilde{\Sigma}(U) = \{W \in \mathcal{U}(k) \mid \exists \text{ a unitary matrix } W_1 \text{ in } V_2 \text{ such that } \|W - UW_1\|_2 \le \frac{3\delta}{\theta}\}.$$

Then

$$\mu(\tilde{\Sigma}(U)) \le (C_1 \cdot 8\delta/\theta)^{k^2} \cdot \left(\frac{C\theta}{\delta}\right)^{mt^2},$$

PROOF. (1) By computing the covering number of the set $\{S \mid S \in V_2, \text{ such that } \|S\|_2 \le 1\}$ by δ/θ-$\|\cdot\|_2$-balls in $\mathcal{M}_k(\mathbb{C})$, we know

$$\nu_2(\{S \mid S \in V_2, \|S\|_2 \le 1\}, \frac{\delta}{\theta}) \le \left(\frac{C\theta}{\delta}\right)^{\text{real dimension of of } V_2} \le \left(\frac{C\theta}{\delta}\right)^{2mt^2+4m(n-m)t+2(n-m)^2},$$

where C is a universal constant. Thus the covering number of the set $\Sigma(U)$ by the $4\delta/\theta$-$\|\cdot\|_2$-balls in $\mathcal{M}_k(\mathbb{C})$ is bounded by

$$\nu_2(\Sigma(U), \frac{4\delta}{\theta}) \le \nu_2(\{S \mid S \in V_2, \|S\|_2 \le 1\}, \frac{\delta}{\theta}) \le \left(\frac{C\theta}{\delta}\right)^{2mt^2+4m(n-m)t+2(n-m)^2}.$$

But the ball of radius $4\delta/\theta$ in $\mathcal{U}(k)$ has the volume bounded by

$$\mu(\text{ball of radius } 4\delta/\theta) \le (C_1 \cdot 4\delta/\theta)^{k^2},$$

where C_1 is a universal constant. Thus

$$\mu(\Sigma(U)) \le (C_1 \cdot 4\delta/\theta)^{k^2} \cdot \left(\frac{C\theta}{\delta}\right)^{2mt^2+4m(n-m)t+2(n-m)^2}.$$

(2) A slight adaption of the proof of part (1) gives us the proof of part (2). □

LEMMA 3.3. *Let $\Omega(A)$ be defined as in (vi) at the beginning of this subsection.*

(1) *The covering number of* $\Omega(A)$ *by the* $\frac{1}{2}\delta$-$\|\cdot\|_2$-*balls in* $\mathcal{M}_k(\mathbb{C})$ *is bounded below by*

$$\nu_2(\Omega(A), \frac{1}{2}\delta) \geq (C_1 \cdot 4\delta/\theta)^{-k^2} \cdot \left(\frac{C\theta}{\delta}\right)^{-(2mt^2+4m(n-m)t+2(n-m)^2)}$$

(2) *If* $n = m$*, then*

$$\nu_2(\Omega(A), \frac{1}{2}\delta) \geq (C_1 \cdot 8\delta/\theta)^{-k^2} \cdot \left(\frac{C\theta}{\delta}\right)^{-mt^2}.$$

PROOF. (1) For every $U \in \mathcal{U}(k)$, define

$$\Sigma(U) = \{W \in \mathcal{U}(k) \mid \exists\ S = S^* \in V_1, \text{ such that } \|S\|_2 \leq 1,\ \|W - US\|_2 \leq \frac{\delta}{\theta}\}.$$

By preceding lemma, we have

$$\mu(\Sigma(U)) \leq (C_1 \cdot 4\delta/\theta)^{k^2} \cdot \left(\frac{C\theta}{\delta}\right)^{mt^2+4m(n-m)t+2(n-m)^2}.$$

A "parking" (or exhausting) argument will show the existence of a family of unitary elements $\{U_i\}_{i=1}^N \subset \mathcal{U}(k)$ such that

$$N \geq (C_1 \cdot 4\delta/\theta)^{-k^2} \cdot \left(\frac{C\theta}{\delta}\right)^{-(mt^2+4m(n-m)t+2(n-m)^2)}$$

and

$$U_i \text{ is not contained in } \cup_{j=1}^{i-1} \Sigma(U_j), \qquad \forall\ i = 1, \ldots, N.$$

From the definition of each $\Sigma(U_j)$, it follows that

$$\|U_i - U_jS\|_2 > \frac{\delta}{\theta}, \qquad \forall\ S \in V_2, \text{ with } \|S\|_2 \leq 1,\ \forall 1 \leq j < i \leq N.$$

By Lemma 3.1, we know that

$$\|U_iAU_i^* - U_jAU_j^*\|_2 > \delta, \qquad \forall 1 \leq j < i \leq N,$$

which implies that

$$\nu_2(\Omega(A), \frac{1}{2}\delta) \geq N \geq (C_1 \cdot 4\delta/\theta)^{-k^2} \cdot \left(\frac{C\theta}{\delta}\right)^{-(mt^2+4m(n-m)t+2(n-m)^2)}$$

(2) is similar as (1). ∎

3.3. We have following theorem.

Theorem 3.2. *Let $n \geq m$, $\delta, \theta > 0$ and $\{\lambda_1, \lambda_2, \ldots, \lambda_m\} \cup \{\lambda_{m+1}, \ldots, \lambda_n\}$ be a family of real numbers such that*

$$|\lambda_i - \lambda_j| \geq \theta$$

for all $1 \leq i < j \leq m$. Let k be a positive integer such that $k - (n-m)$ is divided by m and

$$t = \frac{k-n+m}{m}.$$

Let

$$B = diag(\lambda_{m+1}, \ldots, \lambda_n)$$

be a diagonal matrix in $\mathcal{M}_{n-m}(\mathbb{C})$ and

$$A = diag(\lambda_1 I_t, \lambda_2 I_t, \ldots, \lambda_m I_t, B)$$

be a block-diagonal matrix in $\mathcal{M}_k(\mathbb{C})$, where I_t is the identity matrix in $\mathcal{M}_t(\mathbb{C})$. We let

$$\Omega(A) = \{U^*AU \mid U \text{ is in } \mathcal{U}(k)\}.$$

Then the covering number of $\Omega(A)$ by the $\frac{1}{2}\delta$-$\|\cdot\|$-balls in $\mathcal{M}_k(\mathbb{C})$ is bounded below by

$$\nu_\infty(\Omega(A), \frac{1}{2}\delta) \geq (C_1 \cdot 4\delta/\theta)^{-k^2} \cdot \left(\frac{C\theta}{\delta}\right)^{-(2mt^2+4m(n-m)t+2(n-m)^2)},$$

where C, C_1 are some universal constants.

When $n = m$, we have

$$\nu_\infty(\Omega(A), \frac{1}{2}\delta) \geq (C_1 \cdot 8\delta/\theta)^{-k^2} \cdot \left(\frac{C\theta}{\delta}\right)^{-mt^2},$$

Proof. Note that

$$\nu_\infty(\Omega(A), \frac{\delta}{2}) \geq \nu_2(\Omega(A), \frac{\delta}{2}), \qquad \forall\ \delta > 0.$$

The result follows directly from preceding lemma. ∎

3.4. The following proposition, whose proof is skipped, is an easy extension of Lemma 3.3.

Proposition 3.1. *Let m, k be some positive integers and θ, δ be some positive numbers. Let $T_1, T_2, \ldots, T_{m+1}$ is a partition of the set $\{1, 2, \ldots, k\}$, i.e. $\cup_{i=1}^{m+1} T_i = \{1, 2, \ldots, k\}$ and $T_i \cap T_j = \emptyset$ for $1 \leq i \neq j \leq m+1$. Let $\lambda_1, \ldots, \lambda_k$ be some real numbers such that, if $1 \leq j_1 \neq j_2 \leq m$ then*

$$|\lambda_{i_1} - \lambda_{i_2}| > \theta, \qquad \forall\ i_1 \in T_{j_1},\ i_2 \in T_{j_2}.$$

Let $A = diag(\lambda_1, \lambda_2, \ldots, \lambda_k)$ be a self-adjoint matrix in $\mathcal{M}_k(\mathbb{C})$ and

$$\Omega(A) = \{U^*AU \mid U \in \mathcal{U}(k)\}$$

be a subset of $\mathcal{M}_k(\mathbb{C})$.

Let s_j be the cardinality of the set T_j for $1 \le j \le m+1$. Then the covering number of $\Omega(A)$ by the $\frac{1}{2}\delta$-$\|\cdot\|_2$-balls in $\mathcal{M}_k(\mathbb{C})$ is bounded below by

$$\nu_2(\Omega(A), \frac{1}{2}\delta) \ge (C_1 \cdot 4\delta/\theta)^{-k^2} \cdot \left(\frac{C\theta}{\delta}\right)^{-2s_1^2-2s_2^2-\cdots-2s_{m+1}^2-4(s_1+\cdots+s_m)s_{m+1}},$$

where C, C_1 are some universal constants.

4. Topological free entropy dimension of one variable

Suppose x is a self-adjoint element of a unital C* algebra $\mathcal{A}$. In this section, we are going to compute the topological entropy dimension of x.

4.1. Upperbound.

Proposition 4.1. *Suppose x in $\mathcal{A}$ is a self-adjoint element with the spectrum $\sigma(x)$. Then*

$$\delta_{top}(x) \le 1 - \frac{1}{n},$$

where n is the cardinality of $\sigma(x)$. Here we assume that $\frac{1}{\infty} = 0$.

Proof. By [**26**], we know that the inequality always holds when n is infinity. We need only to show that

$$\delta_{top}(x) \le 1 - \frac{1}{n},$$

when $n < \infty$.

Assume that $\lambda_1, \ldots, \lambda_n$ are in the spectrum of x in $\mathcal{A}$.

Let $R > \|x\|$. By Theorem 3.1, for every $\omega > 0$, there are $r_0 > 0$ and $\epsilon_0 > 0$ such that, for all $r > r_0, \epsilon < \epsilon_0$,

$$A \in \Gamma_R^{(top)}(x; k, \epsilon, P_1, \ldots, P_r),$$

there are some $1 \le k_1, \ldots, k_n \le k$, with $k_1 + \cdots + k_n = k$ and a unitary matrix U in $\mathcal{M}_k(\mathbb{C})$ satisfying

$$\left\| A - U \begin{pmatrix} \lambda_1 I_{k_1} & 0 & \cdots & 0 \\ 0 & \lambda_2 I_{k_2} & \cdots & 0 \\ 0 & 0 & \cdots & \lambda_n I_{k_n} \end{pmatrix} U^* \right\| \le 2\omega. \qquad (**)$$

Let

$$\Omega(k_1, \ldots, k_n) = \left\{ U \begin{pmatrix} \lambda_1 I_{k_1} & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2 I_{k_2} & \cdots & 0 & 0 \\ 0 & 0 & \cdots & \lambda_{n-1} I_{k_{n-1}} & 0 \\ 0 & 0 & \cdots & 0 & \lambda_n I_{k_n} \end{pmatrix} U^* \mid U \text{ is in } \mathcal{U}_k \right\}.$$

By Corollary 12 in [**21**] or Theorem 3 in [**2**], the covering number of $\Omega(k_1, \ldots, k_{n-1}, k_n)$ by ω-$\|\cdot\|$-balls in $\mathcal{M}_k(\mathbb{C})$ is upperbounded by

$$\nu_\infty(\Omega(k_1, \ldots, k_{n-1}, k_n), \omega) \leq \left(\frac{C_2}{\omega}\right)^{k^2 - \sum_{i=1}^n k_i^2},$$

where C_2 is a constant which does not depend on $k, k_1, \ldots, k_n$ (may depend on n and $\|x\|$).

Let $\mathcal{I}$ be the set consisting of all these $(k_1, \ldots, k_n)$ in $\mathbb{Z}^n$ such that $1 \leq k_1, \ldots, k_n \leq k$ and $k_1 + \cdots + k_n = k$. Then the cardinality of the set $\mathcal{I}$ is equal to

$$\frac{(k-1)!}{(n-1)!(k-n)!}.$$

Note that

$$\sum_{i=1}^n k_i^2 \geq k^2/n$$

for all $1 \leq k_1, \ldots, k_n \leq k$ with $k_1 + \cdots + k_n = k$; and by $(**)$

$$\Gamma_R^{(top)}(x; k, \epsilon, P_1, \ldots, P_r)$$

is contained in 2ω-neighborhood of the set

$$\bigcup_{(k_1,\ldots,k_n)\in\mathcal{I}} \Omega(k_1, \ldots, k_n).$$

It follows that the covering number of the set

$$\Gamma_R^{(top)}(x; k, \epsilon, P_1, \ldots, P_r)$$

by 3ω-$\|\cdot\|$-balls in $\mathcal{M}_k(\mathbb{C})$ is upperbounded by

$$\nu_\infty(\Gamma_R^{(top)}(x; k, \epsilon, P_1, \ldots, P_r), 3\omega) \leq \frac{(k-1)!}{(n-1)!(k-n)!} \cdot \left(\frac{C_2}{\omega}\right)^{k^2 - k^2/n}.$$

Thus

$$\delta_{top}(x) \leq \limsup_{\omega\to 0^+} \limsup_{k\to\infty} \frac{\log\left(\frac{(k-1)!}{(n-1)!(k-n)!} \cdot \left(\frac{C_2}{\omega}\right)^{k^2-k^2/n}\right)}{-k^2\log(3\omega)} = 1 - \frac{1}{n}.$$

□

4.2. Lower-bound. We follow the notation from last subsection.

PROPOSITION 4.2. *Suppose that x is a self-adjoint element with the finite spectrum $\sigma(x)$ in $\mathcal{A}$. Then*

$$\delta_{top}(x) \geq 1 - \frac{1}{n},$$

where n is the cardinality of the set $\sigma(x)$.

PROOF. Suppose that $\lambda_1, \ldots, \lambda_n$ are distinct spectrum of x. There is some positive number θ such that

$$|\lambda_i - \lambda_j| > \theta, \qquad \forall \ \ 1 \le i \ne j \le n.$$

Assume $k = nt$ for some positive integer t. Let

$$A_k = diag(\lambda_1 I_t, \ldots, \lambda_n I_t)$$

be a diagonal matrix in $\mathcal{M}_k(\mathbb{C})$ where I_t is the $t \times t$ identity matrix. It is easy to see that, for all $R > \|x\|$, $r \ge 1$ and $\epsilon > 0$, we have

$$A_k \in \Gamma_R^{(top)}(x; k, \epsilon, P_1, \ldots, P_r).$$

For any $\omega > 0$, applying Theorem 3.2 for $n = m$ and $\delta = \frac{1}{2}\omega$, we have

$$\begin{aligned}\nu_\infty(\Gamma_R^{(top)}(x; k, \epsilon, P_1, \ldots, P_r), \omega) &\ge (C_1 \cdot 8\delta/\theta)^{-k^2} \cdot \left(\frac{C\theta}{\delta}\right)^{-mt^2} \\ &= (16 C_1 \omega/\theta)^{-k^2} \cdot \left(\frac{2C\theta}{\omega}\right)^{-mt^2}.\end{aligned}$$

Note that $k = nt = mt$ and θ is some fixed number. A quick computation shows that

$$\delta_{top}(x) \ge 1 - \frac{1}{n}.$$

$\square$

PROPOSITION 4.3. *Suppose that x is a self-adjoint element in $\mathcal{A}$ with infinite spectrum. Then*

$$\delta_{top}(x) \ge 1.$$

PROOF. For any $0 < \theta < 1$, there are $\lambda_1, \ldots, \lambda_m$ in the spectrum of x, $\sigma(x)$, satisfying (i)

$$|\lambda_i - \lambda_j| \ge \theta;$$

and (ii) for any λ in $\sigma(x)$, there is some λ_j with $|\lambda - \lambda_j| \le \theta$. By functional calculus, for any $R > \|x\|$, $r \ge 1$ and $\epsilon > 0$, there are some positive integer $n \ge m$ and real numbers $\lambda_{m+1}, \ldots, \lambda_n$ in $\sigma(x)$ satisfying: for every $t \ge 1$ the matrix

$$A = diag(\lambda_1 I_t, \lambda_2 I_t, \ldots, \lambda_m I_t, \lambda_{m+1}, \ldots, \lambda_n)$$

is in

$$\Gamma_R^{(top)}(x; k, \epsilon, P_1, \ldots, P_r),$$

where we assume that $k = mt + n - m$. For any $\omega > 0$, let $\delta = \frac{1}{2}\omega$. By Theorem 3.2, we know that

$$\nu_\infty(\Gamma_R^{(top)}(x; k, \epsilon, P_1, \ldots, P_r), \omega) \ge (C_1 \cdot 4\delta/\theta)^{-k^2} \cdot \left(\frac{C\theta}{\delta}\right)^{-(2mt^2 + 4m(n-m)t + 2(n-m)^2)}.$$

Thus

$$\limsup_{k \to \infty} \frac{\log(\nu_\infty(\Gamma_R^{(top)}(x; k, \epsilon, P_1, \ldots, P_r), \omega))}{-k^2 \log \omega} \ge \frac{\log(\frac{4C_1}{2}) - \log\theta}{\log\omega} + 1 + \frac{2}{m}\frac{\log(2C) + \log\theta}{\log\omega} - \frac{2}{m}.$$

Then,

$$\delta_{top}(x) \geq 1 - \frac{2}{m}.$$

When θ goes to 0, m goes to infinity as $\sigma(x)$ has infinitely many elements. Therefore,

$$\delta_{top}(x) \geq 1.$$

□

4.3. Topological free entropy dimension in one variable case. By Proposition 4.1, Proposition 4.2 and Proposition 4.3, we have the following result.

Theorem 4.1. *Suppose x is a self-adjoint element in a unital C^* algebra A. Then*

$$\delta_{top}(x) = 1 - \frac{1}{n},$$

where n is the cardinality of the set $\sigma(x)$ and $\sigma(x)$ is the set of spectrum of x in $\mathcal{A}$. Here we assume that $\frac{1}{\infty} = 0$.

5. Topological free entropy dimension of n-tuple in unital C* algebras

5.1. An equivalent definition of topological free entropy dimension. Suppose that $\mathcal{A}$ is a unital C* algebra and $x_1, \ldots, x_n, y_1, \ldots, y_m$ are self-adjoint elements in $\mathcal{A}$. For every $R, \epsilon > 0$ and positive integers r, k, let

$$\Gamma_R^{(top)}(x_1, \ldots, x_n : y_1, \ldots, y_m; k, \epsilon, P_1, \ldots, P_r)$$

be Voiculescu's norm-microstate space defined in section 2.4.

Define

$$\nu_2(\Gamma_R^{(top)}(x_1, \ldots, x_n : y_1, \ldots, y_m; k, \epsilon, P_1, \ldots, P_r), \omega)$$

to be the covering number of the set $\Gamma_R^{(top)}(x_1, \ldots, x_n : y_1, \ldots, y_m; k, \epsilon, P_1, \ldots, P_r)$ by ω-$\|\cdot\|_2$-balls in the metric space $(\mathcal{M}_k^{s.a}(\mathbb{C}))^n$ equipped with trace norm (see Definition 2.2).

Definition 5.1. *Define*

$$\begin{aligned}\tilde{\delta}_{top}(x_1, \ldots, &\ x_n : y_1, \ldots, y_m; \omega) \\ &= \sup_{R>0} \inf_{\epsilon>0, r\in\mathbb{N}} \limsup_{k\to\infty} \frac{\log(\nu_2(\Gamma_R^{(top)}(x_1, \ldots, x_n : y_1, \ldots, y_m; k, \epsilon, P_1, \ldots, P_r), \omega))}{-k^2 \log \omega}\end{aligned}$$

And

$$\tilde{\delta}_{top}(x_1, \ldots, \ x_n : y_1, \ldots, y_m) = \limsup_{\omega\to 0^+} \tilde{\delta}_{top}(x_1, \ldots, x_n : y_1, \ldots, y_m; \omega)$$

The following proposition was pointed out by Voiculescu in [**26**]. For the sake of completeness, we also include a proof here.

PROPOSITION 5.1. *Suppose that $\mathcal{A}$ is a unital C^* algebra and $x_1,\ldots,x_n,y_1,\ldots,y_m$ are self-adjoint elements in $\mathcal{A}$. Then*

$$\tilde{\delta}_{top}(x_1,\ldots,\ x_n : y_1,\ldots,y_m) = \delta_{top}(x_1,\ldots,\ x_n : y_1,\ldots,y_m),$$

where $\delta_{top}(x_1,\ldots,\ x_n : y_1,\ldots,y_m)$ is the topological free entropy dimension of $x_1,\ldots,x_n$ in presence of $y_1,\ldots,y_m$.

PROOF. This is an easy consequence of Lemma 1 in [**21**]. Let λ be the Lebesgue measure on $(\mathcal{M}_k^{s.a}(\mathbb{C}))^n$. Let, for every $\omega > 0$,

$$B_\infty(\omega) = \{(A_1,\ldots,A_n) \in (\mathcal{M}_k^{s.a}(\mathbb{C}))^n \mid \|(A_1,\ldots,A_n)\| \le \omega\}$$
$$B_2(\omega) = \{(A_1,\ldots,A_n) \in (\mathcal{M}_k^{s.a}(\mathbb{C}))^n \mid \|(A_1,\ldots,A_n)\|_2 \le \omega\}$$

It follows from the results in [**21**] or Theorem 8 in [**2**] that, for some M_1, M_2 independent of k,ω such that

$$\lambda(B_\infty(1)) \le \lambda(B_\infty(\omega/4))\left(\frac{M_1}{\omega}\right)^{nk^2} \quad \text{and} \quad \left(\frac{M_2}{2\sqrt{n}\omega}\right)^{nk^2}\lambda(B_2(2\sqrt{n}\omega)) \le \lambda(B_2(1)). \quad (5.1.1)$$

For every $\omega > 0$ and any subset set K of $(\mathcal{M}_k^{s.a}(\mathbb{C}))^n$, let

$$K(\omega,\|\cdot\|) = \{(A_1,\ldots,A_n) \in (\mathcal{M}_k^{s.a}(\mathbb{C}))^n \mid \|(A_1,\ldots,A_n)-(D_1,\ldots,D_n)\| \le \omega \text{ for some } (D_1,\ldots,D_n)\in K\}$$
$$K(\omega,\|\cdot\|_2) = \{(A_1,\ldots,A_n) \in (\mathcal{M}_k^{s.a}(\mathbb{C}))^n \mid \|(A_1,\ldots,A_n)-(D_1,\ldots,D_n)\|_2 \le \omega \text{ for some } (D_1,\ldots,D_n)\in K\}$$

Note the following fact:

$$\|(A_1,\ldots,A_n)\|_2 \le \sqrt{n}\|(A_1,\ldots,A_n)\|, \qquad \forall\ (A_1,\ldots,A_n)\in(\mathcal{M}_k^{s.a}(\mathbb{C}))^n.$$

It follows from Lemma 1 in [**21**] that

$$\nu_\infty(K,\omega) \le \frac{\lambda(K(\omega,\|\cdot\|))}{\lambda(B_\infty(\omega/4))};$$

and

$$\frac{\lambda(K(\sqrt{n}\omega,\|\cdot\|_2))}{\lambda(B_2(2\sqrt{n}\omega))} \le \nu_2(K(\sqrt{n}\omega,\|\cdot\|_2), 2\sqrt{n}\omega).$$

Combining with the equalities (5.1.1), we get

$$\nu_\infty(K,\omega) \le \frac{\lambda(K(\omega,\|\cdot\|))}{\lambda(B_\infty(\omega/4))} \le \left(\frac{M_1}{\omega}\right)^{nk^2}\frac{\lambda(K(\omega,\|\cdot\|))}{\lambda(B_\infty(1))} \le \left(\frac{M_1}{\omega}\right)^{nk^2}\frac{\lambda(K(\sqrt{n}\omega,\|\cdot\|_2))}{\lambda(B_\infty(1))};$$

and

$$\left(\frac{M_2}{2\sqrt{n}\omega}\right)^{nk^2}\frac{\lambda(K(\sqrt{n}\omega,\|\cdot\|_2))}{\lambda(B_2(1))} \le \frac{\lambda(K(\sqrt{n}\omega,\|\cdot\|_2))}{\lambda(B_2(2\sqrt{n}\omega))} \le \nu_2(K(\sqrt{n}\omega,\|\cdot\|_2),2\sqrt{n}\omega) \le \nu_2(K,\sqrt{n}\omega).$$

Therefore, we have

$$\nu_2(K,\sqrt{n}\omega) \le \nu_\infty(K,\omega) \le \left(\frac{2\sqrt{n}M_1}{M_2}\right)^{nk^2} \frac{\lambda(B_2(1))}{\lambda(B_\infty(1))}\nu_2(K,\sqrt{n}\omega).$$

It is a well-known fact (for example see Theorem 8 in [**2**]) that

$$\frac{\lambda(B_2(1))}{\lambda(B_\infty(1))} \le C_3^{nk^2}$$

for some universal constant $C_3 > 0$. Hence

$$\nu_2(K,\sqrt{n}\omega) \le \nu_\infty(K,\omega) \le \left(\frac{2\sqrt{n}M_1C_3}{M_2}\right)^{nk^2} \nu_2(K,\sqrt{n}\omega).$$

Let K be $\Gamma_R^{(top)}(x_1,\ldots,x_n : y_1,\ldots,y_m; k,\epsilon,P_1,\ldots,P_r)$. By the definitions of $\tilde{\delta}_{top}$ and δ_{top}, we have

$$\tilde{\delta}_{top}(x_1,\ldots,\ x_n : y_1,\ldots,y_m) = \delta_{top}(x_1,\ldots,\ x_n : y_1,\ldots,y_m).$$

□

5.2. Upper-bound of topological free entropy dimension in a unital C* algebra. Let us recall Voiculescu's definition of free dimension capacity in [**26**].

Definition 5.2. *Suppose that $\mathcal{A}$ is a unital C* algebra with a family of self-adjoint generators $x_1,\ldots,x_n$. Suppose that $TS(\mathcal{A})$ is the set consisting of all tracial states of $\mathcal{A}$. If $TS(\mathcal{A}) \ne \varnothing$, define* Voiculescu's free dimension capacity $\kappa\delta(x_1,\ldots,x_n)$ *of $x_1,\ldots,x_n$ as follows,*

$$\kappa\delta(x_1,\ldots,x_n) = \sup_{\tau\in TS(\mathcal{A})} \delta_0(x_1,\ldots,x_n : \tau),$$

where $\delta_0(x_1,\ldots,x_n : \tau)$ is Voiculescu's (von Neumann algebra) free entropy dimension of $x_1,\ldots,x_n$ in $\langle\mathcal{A},\tau\rangle$.

The relationship between topological free entropy dimension of a unital C* algebra with a unique tracial state and its free dimension capacity is indicated by the following result.

Theorem 5.1. *Suppose that $\mathcal{A}$ is a unital C* algebra with a family of self-adjoint generators $x_1,\ldots,x_n$. Suppose that $TS(\mathcal{A})$ is the set consisting of all tracial states of $\mathcal{A}$. If $TS(\mathcal{A})$ is a set with a single element, then*

$$\delta_{top}(x_1,\ldots,x_n) \le \kappa\delta(x_1,\ldots,x_n).$$

To prove the preceding theorem, we need the following lemma.

Sublemma 5.2.1. *Suppose that $\mathcal{A}$ is a unital C* algebra with a family of self-adjoint generators $x_1,\ldots,x_n$. Suppose that $TS(\mathcal{A}) \ne \varnothing$ is the set consisting of all tracial states of $\mathcal{A}$. Let $R > \max\{\|x_1\|,\ldots,\|x_n\|\}$ be some positive number. Then for any $m \ge 1$, there is some $r_m \in \mathbb{N}$ such that*

$$\Gamma_R^{(top)}(x_1,\ldots,x_n; k, \frac{1}{r_m}, P_1,\ldots,P_{r_m}) \subseteq \cup_{\tau\in TS(\mathcal{A})}\Gamma_R(x_1,\ldots,x_n; k, m, \frac{1}{m}; \tau), \quad \forall\ k \ge 1$$

where $\Gamma_R(x_1,\ldots,x_n;k,m,\frac{1}{m};\tau)$ *is microstate space of* $x_1,\ldots,x_n$ *with respect to* τ *(see* [**23**]*).*

Proof of Sublemma 5.2.1: We will prove the result by contradiction. Suppose, to the contrary, there is some $m_0 \geq 1$ so that following holds: for any $r \in \mathbb{N}$, there are some $k_r \geq 1$ and some

$$(A_1^{(r)}, A_2^{(r)}, \ldots, A_n^{(r)}) \in \Gamma_R^{(top)}(x_1,\ldots,x_n;k_r,\frac{1}{r},P_1,\ldots,P_r)$$

satisfying

$$(A_1^{(r)}, A_2^{(r)}, \ldots, A_n^{(r)}) \notin \cup_{\tau\in TS(\mathcal{A})}\Gamma_R(x_1,\ldots,x_n;k_r,m,\frac{1}{m};\tau). \tag{5.2.1}$$

Let α be a free ultrafilter in $\beta(\mathbb{N})\backslash\mathbb{N}$. Let $\mathcal{N} = \prod_{r=1}^{\alpha}\mathcal{M}_{k_r}(\mathbb{C})$ be the von Neumann algebra ultraproduct of $\{\mathcal{M}_{k_r}(\mathbb{C})\}_{r=1}^{\infty}$ along the ultrafilter α, i.e. $\prod_{r=1}^{\alpha}\mathcal{M}_{k_r}(\mathbb{C})$ is the quotient algebra of the C* algebra $\prod_{r=1}^{\infty}\mathcal{M}_{k_r}(\mathbb{C})$ by $\mathcal{I}_2$, the 0-ideal of the trace τ_α, where $\tau_\alpha((A_r)_{r=1}^{\infty}) = \lim_{r\to\alpha}\frac{Tr(A_r)}{k_r}$. Let, for each $1 \leq j \leq n$, $a_j = [(A_r^{(j)})_{r=1}^{\infty}]$ be a self-adjoint element in $\mathcal{N}$. By mapping x_j to a_j, there is a unital $*$-homomorphism ψ from the C* algebra $\mathcal{A}$ onto the C* subalgebra generated by $\{a_1,\ldots,a_n\}$ in $\mathcal{N}$.

Let τ_0 be the tracial state on $\mathcal{A}$ which is induced by τ_α on $\psi(\mathcal{A})$, i.e.

$$\tau_0(x) = \tau_\alpha(\psi(x)), \qquad \forall\ x \in \mathcal{A}.$$

It follows that when r is large enough,

$$(A_1^{(r)}, A_2^{(r)}, \ldots, A_n^{(r)}) \in \Gamma_R(x_1,\ldots,x_n;k_r,m,\frac{1}{m};\tau_0),$$

which contradicts with the inequality (5.2.1). This complete the proof. $\square$

Proof of Theorem 5.1: Let $R > \max\{\|x_1\|,\ldots,\|x_n\|\}$. Let τ be the unique trace of $\mathcal{A}$. By Sublemma 5.2.1, for any $m \geq 1$, there is $r \in \mathbb{N}$ such that

$$\Gamma_R^{(top)}(x_1,\ldots,x_n;k,\frac{1}{r},P_1,\ldots,P_r) \subseteq \Gamma_R(x_1,\ldots,x_n;k,m,\frac{1}{m};\tau), \quad \forall\ k \geq 1.$$

Therefore, for any $1 > \omega > 0$, we have

$$\nu_2(\Gamma_R^{(top)}(x_1,\ldots,x_n;k,\frac{1}{r},P_1,\ldots,P_r),\omega) \leq \nu_2(\Gamma_R(x_1,\ldots,x_n;k,m,\frac{1}{m};\tau),\omega), \quad \forall\ k \geq 1.$$

Now it is easy to check that

$$\tilde{\delta}_{top}(x_1,\ldots,x_n) \leq \delta(x_1,\ldots,x_n;\tau) = \kappa\delta(x_1,\ldots,x_n).$$

By Proposition 5.1, we know that

$$\delta_{top}(x_1,\ldots,x_n) \leq \kappa\delta(x_1,\ldots,x_n).$$

$\square$

Remark 5.1. *Combining Theorem 5.1 with the results in* [**11**] *or* [**14**]*, we will be able to compute the upper-bound of topological free entropy dimension for a large class of unital C* algebras. For example,* $\delta_{top}(x_1,\ldots,x_n) \leq 1$ *if* $x_1,\ldots,x_n$ *is a family of self-adjoint operators that generates an irrational rotation algebra* $\mathcal{A}$.

5.3. Lower-bound of topological free entropy dimension in a unital C* algebra. In this subsection, we assume that $\mathcal{A}$ is a finitely generated, infinite dimensional, unital simple C* algebra with a unique tracial state τ. Assume that $x_1, \ldots, x_n$ is a family of self-adjoint generators of $\mathcal{A}$. Let H be the Hilbert space $L^2(\mathcal{A}, \tau)$. Without loss of generality, we might assume that $\mathcal{A}$ is faithfully represented on the Hilbert space H. Let $\mathcal{M}$ be the von Neumann algebra generated by $\mathcal{A}$ on H. It is not hard to see that $\mathcal{M}$ is a diffuse von Neumann algebra with a tracial state τ.

For each positive integer m, there is a family of mutually orthogonal projections $p_1, \ldots, p_m$ in $\mathcal{M}$ such that $\tau(p_j) = 1/m$ for $1 \le j \le m$. Let

$$y_m = 1 \cdot p_1 + 2 \cdot p_2 + \cdots + m \cdot p_m = \sum_{j=1}^{m} j \cdot p_j \ \in \mathcal{M}.$$

Let $\{P_r(x_1, \ldots, x_n)\}_{r=1}^{\infty}$ be defined as in section 2.3. Thus $\{P_r(x_1, \ldots, x_n)\}_{r=1}^{\infty}$ is dense in $\mathcal{M}$ with respect to the strong operator topology. Hence, for each $m \ge 1$, there is some self-adjoint element $P_{r_m}(x_1, \ldots, x_n)$ in $\mathcal{A}$ such that

$$\|y_m - P_{r_m}(x_1, \ldots, x_n)\|_2 \le \frac{1}{m^3},$$

where $\|a\|_2 = \sqrt{\tau(a^* a)}$ for all $a \in \mathcal{M}$.

Lemma 5.1. *Let $\mathcal{A}$ be finitely generated, infinite dimensional, unital simple C* algebra with a unique tracial state τ. Assume that $x_1, \ldots, x_n$ is a family of self-adjoint generators of $\mathcal{A}$. Let H, $\mathcal{M}$ be defined as above. For each $m \ge 1$, let y_m and $P_{r_m}(x_1, \ldots, x_n)$ be chosen as above. Then*

$$\delta_{top}(x_1, \ldots, x_n) \ge \delta_{top}(P_{r_m}(x_1, \ldots, x_n) : x_1, \ldots, x_n).$$

Proof. Let $R > \max\{\|P_{r_m}(x_1, \ldots, x_n)\|, \|x_1\|, \ldots, \|x_n\|\}$. There exists a positive constant $D > 1$ such that

$$\|P_{r_m}(A_1, \ldots, A_n) - P_{r_m}(B_1, \ldots, B_m)\| \le D\|(A_1, \ldots, A_n) - (B_1, \ldots, B_m)\|$$

for all $A_1, \ldots, A_n, B_1, \ldots, B_n$ in $\mathcal{M}_k(\mathbb{C})$ satisfying $0 \le \|A_1\|, \ldots, \|A_n\|, \|B_1\|, \ldots, \|B_n\| \le R$.

Then it is not hard to verify that, for $\omega > 0$,

$$\begin{aligned} &\nu_\infty(\Gamma_R^{(top)}(P_{r_m}(x_1, \ldots, x_n) : x_1, \ldots, x_n; k, \epsilon, P_1, \ldots, P_r), \omega) \\ &\qquad \le \nu_\infty(\Gamma_R^{(top)}(x_1, \ldots, x_n; k, \epsilon, P_1, \ldots, P_r), \frac{\omega}{4D}) \end{aligned}$$

for each $r \ge r_m$ and $\epsilon < \frac{\omega}{4}$. By definition of δ_{top} and Remark 2.3, we have

$$\delta_{top}(P_{r_m}(x_1, \ldots, x_n) : x_1, \ldots, x_n) \le \delta_{top}(x_1, \ldots, x_n).$$

$\square$

Definition 5.3. *Suppose $\mathcal{A}$ is a unital C^* algebra and $x_1, \ldots, x_n$ is a family of self-adjoint elements of $\mathcal{A}$ that generates $\mathcal{A}$ as a C^* algebra. If for any $R > \max\{\|x_1\|, \ldots, \|x_n\|, \|y_1\|, \ldots, \|y_m\|\}$, $r > 0$, $\epsilon > 0$, there is a sequence of positive integers $k_1 < k_2 < \cdots$ such that*

$$\Gamma_R^{(top)}(x_1, \ldots, x_n : y_1, \ldots, y_m; k_s, \epsilon, P_1, \ldots, P_r) \neq \varnothing, \qquad \forall\ s \geq 1$$

then $\mathcal{A}$ is called having approximation property.

Lemma 5.2. *Let $\mathcal{A}$ be a finitely generated, infinite dimensional, unital simple C^* algebra with a unique tracial state τ. Assume that A has approximation property. Assume that $x_1, \ldots, x_n$ is a family of self-adjoint generators of $\mathcal{A}$. Let H, $\mathcal{M}$ be defined as above. Let m be a positive integer. Let y_m and $P_{r_m}(x_1, \ldots, x_n)$ be chosen as above. Let $R > \max\{\|P_{r_m}(x_1, \ldots, x_n)\|, \|x_1\|, \ldots, \|x_n\|\}$. Then there is some positive integer $r > r_m$ so that the following hold: $\forall\ k \geq 1$, if*

$$(B, A_1, \ldots, A_n) \in \Gamma_R^{(top)}(P_{r_m}(x_1, \ldots, x_n), x_1, \ldots, x_n; k, \frac{1}{r}, P_1, \ldots, P_r),$$

then there are some $1 \leq k_1, \ldots, k_m \leq k$ with $\frac{1}{m} - \frac{1}{r} \leq \frac{k_j}{k} \leq \frac{1}{m} + \frac{1}{r}$ for each $1 \leq j \leq m$ and $k_1 + \cdots + k_m = k$, and a unitary matrix U in $\mathcal{U}(k)$ satisfying

$$\|B - U \begin{pmatrix} 1 \cdot I_{k_1} & 0 & \cdots & 0 \\ 0 & 2 \cdot I_{k_2} & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & \cdots & m \cdot I_{k_m} \end{pmatrix} U^*\|_2 \leq \frac{2}{m^3}.$$

Proof. We will prove the result by contradiction. Assume, to the contrary, for all $r \geq r_m$ there are some $k_r \geq 1$ and some

$$(B^{(r)}, A_1^{(r)}, \ldots, A_n^{(r)}) \in \Gamma_R^{(top)}(P_{r_m}(x_1, \ldots, x_n), x_1, \ldots, x_n; k_r, \frac{1}{r}, P_1, \ldots, P_r),$$

satisfying

$$\|B^{(r)} - U \begin{pmatrix} 1 \cdot I_{s_1} & 0 & \cdots & 0 \\ 0 & 2 \cdot I_{s_2} & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & \cdots & m \cdot I_{s_m} \end{pmatrix} U^*\|_2 > \frac{2}{m^3}, \tag{5.3.1}$$

for all $1 \leq s_1, \ldots, s_m \leq k_r$ with $\frac{1}{m} - \frac{1}{r} \leq \frac{s_j}{k_r} \leq \frac{1}{m} + \frac{1}{r}$ for each $1 \leq j \leq n$ and $s_1 + \cdots + s_m = k_r$, and all unitary matrix U in $\mathcal{U}(k)$.

Let α be a free ultrafilter in $\beta(\mathbb{N}) \setminus \mathbb{N}$. Let $\mathcal{N} = \prod_{r=1}^{\alpha} \mathcal{M}_{k_r}(\mathbb{C})$ be the von Neumann algebra ultraproduct of $\{\mathcal{M}_{k_r}(\mathbb{C})\}_{r=1}^{\infty}$ along the ultrafilter α, i.e. $\prod_{r=1}^{\alpha} \mathcal{M}_{k_r}(\mathbb{C})$ is the quotient of the C* algebra $\prod_{r=1}^{\infty} \mathcal{M}_{k_r}(\mathbb{C})$ by $\mathcal{I}_2$, the 0-ideal of the trace τ_α, where $\tau_\alpha((A_r)_{r=1}^{\infty}) = \lim_{r \to \alpha} \frac{Tr(A_r)}{k_r}$. Let, for each $1 \leq j \leq n$, $a_j = [(A_r^{(j)})_{r=1}^{\infty}]$ be a self-adjoint element in $\mathcal{N}$. By mapping x_j to a_j, there is a unital $*$-homomorphism ψ from the C* algebra $\mathcal{A}$ onto the C* subalgebra generated

by $\{a_1, \ldots, a_n\}$ in $\mathcal{N}$. Since $\mathcal{A}$ is a simple C* algebra and $\psi(I_{\mathcal{A}}) = I_{\mathcal{N}}$, ψ actually is a $*$-isomorphism. Since $\mathcal{A}$ has a unique trace τ, ψ induces a $*$-isomorphism (still denoted by ψ) from $\mathcal{M}$ onto the von Neumann subalgebra generated by $a_1, \ldots, a_n$ in $\mathcal{N}$. Therefore,

$$\|y_m - P_{r_m}(x_1, \ldots, x_n)\|_2 = \|\psi(y_m) - P_{r_m}(a_1, \ldots, a_n)\|_{2,\tau_\alpha} \le \frac{1}{m^3}.$$

This contradicts with the definition of y_m and inequality (5.3.1). $\square$

The following lemma is well-known (for example, see Lemma 4.1 in [**23**]).

Lemma 5.3. *Suppose A, or B, is a self-adjoint matrix in $\mathcal{M}_k^{s.a.}(\mathbb{C})$ with a list of eigenvalues $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_k$, or $\mu_1 \le \mu_2 \le \cdots \le \mu_k$ respectively. Then*

$$\sum_{j=1}^{k} |\lambda_j - \mu_j|^2 \le Tr((A - UBU^*)^2),$$

where U is any unitary matrix in $\mathcal{U}(k)$.

Lemma 5.4. *Let r, m be some positive integer with $4 < m < r$. Suppose $k_1, \ldots, k_m$ is a family of positive integers such that $\frac{1}{m} - \frac{1}{r} \le \frac{k_j}{k} \le \frac{1}{m} + \frac{1}{r}$ for all $1 \le j \le k$ and $k_1 + \cdots + k_m = k$. If A is a self-adjoint matrix in $\mathcal{M}_k(\mathbb{C})$ such that, for some unitary matrix U in $\mathcal{U}(k)$,*

$$\|A - U \begin{pmatrix} 1 \cdot I_{k_1} & 0 & \cdots & 0 \\ 0 & 2 \cdot I_{k_2} & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & \cdots & m \cdot I_{k_m} \end{pmatrix} U^*\|_2 \le \frac{2}{m^3},$$

then, for any $\omega > 0$ we have

$$\nu_2(\Omega(A), \omega) \ge (8C_1\omega)^{-k^2} \cdot \left(\frac{2C}{\omega}\right)^{\frac{-56k^2}{m}}$$

for some constants $C_1, C > 1$ independent of k, ω, where

$$\Omega(A) = \{W^*AW \mid W \in \mathcal{U}(k)\}.$$

Proof. Suppose that $\lambda_1 \le \lambda_2 \le \ldots \le \lambda_k$ are the eigenvalues of A. For each $1 \le j \le m$, let

$$T_j = \{i \in \mathbb{N} \mid (\sum_{t=0}^{j-1} k_t) + 1 \le i \le \sum_{t=0}^{j} k_t \text{ and } |\lambda_i - j| \le \frac{1}{m}\}$$

and

$$\hat{T}_j = \{(\sum_{t=0}^{j-1} k_t) + 1, (\sum_{t=0}^{j-1} k_t) + 2, \cdots, \sum_{t=0}^{j} k_t\} \setminus T_j,$$

here we assume that $k_0 = 0$. Let $B = diag(1 \cdot I_{k_1}, \cdots, m \cdot I_{k_m})$ be a diagonal matrix in $\mathcal{M}_k(\mathbb{C})$. By Lemma 5.3, we have

$$k \left(\frac{2}{m^3}\right)^2 \geq Tr((A - UBU^*)^2) \geq \sum_{i \ \in \hat{T}_j} |\lambda_i - j|^2 \geq \left(\frac{1}{m}\right)^2 \ card(\hat{T}_j),$$

where $card(\hat{T}_j)$ is the cardinality of the set $\hat{T}_j$. Thus

$$card(\hat{T}_j) \leq \frac{4k}{m^4}, \qquad \text{for } 1 \leq j \leq m.$$

Let $s_j = card(T_j)$ for $1 \leq j \leq m$, whence

$$\frac{k}{m} + \frac{k}{r} \geq k_j \geq s_j = k_j - card(\tilde{T}_j) \geq k_j - \frac{4k}{m^4} \geq \frac{k}{m} - \frac{k}{r} - \frac{4k}{m^4}, \qquad \forall \ 1 \leq j \leq m.$$

Let

$$T_{m+1} = \{1, 2, \ldots, k\} \setminus (\cup_{j=1}^n T_j)$$

and s_{m+1} be the cardinality of the set T_{m+1}. Thus

$$s_{m+1} = k - s_1 - \cdots - s_m = \sum_{j=1}^m card(\hat{T}_j) \leq \sum_{j=1}^m \frac{4k}{m^4} = \frac{4k}{m^3}.$$

It is not hard to see that $T_1, \ldots, T_{m+1}$ is a partition of the set $\{1, 2, \ldots, k\}$. Moreover, if $1 \leq j_1 \neq j_2 \leq m$ then for any

$$i_1 \in T_{j_1}, \qquad \text{and} \qquad i_2 \in T_{j_2}$$

we have

$$|\lambda_{i_1} - \lambda_{i_2}| \geq |j_2 - j_1| - |\lambda_{i_2} - j_2| - |\lambda_{i_1} - j_1| \geq 1 - \frac{2}{m} \geq \frac{1}{2}.$$

Applying Proposition 3.1 for such $T_1, \ldots, T_m, T_{m+1}$, $\theta = 1/2$ and $\omega = \delta/2$, we have

$$\begin{aligned}
\nu_2(\Omega(A), \omega) &\geq (8C_1\omega)^{-k^2} \cdot \left(\frac{2C}{\omega}\right)^{-2s_1^2 - \cdots - 2s_m^2 - 2s_{m+1}^2 - 4(s_1 + \cdots + s_m)s_{m+1}} \\
&\geq (8C_1\omega)^{-k^2} \cdot \left(\frac{2C}{\omega}\right)^{-2(k_1^2 + \cdots + k_m^2 + (\frac{4k}{m^3})^2 + 2k \cdot \frac{4k}{m^3})} \\
&\geq (8C_1\omega)^{-k^2} \cdot \left(\frac{2C}{\omega}\right)^{-2((\frac{k}{m} + \frac{k}{r})^2 + \cdots + (\frac{k}{m} + \frac{k}{r})^2 + \frac{16k^2}{m^6} + \frac{8k^2}{m^3})} \\
&\geq (8C_1\omega)^{-k^2} \cdot \left(\frac{2C}{\omega}\right)^{\frac{-56k^2}{m}},
\end{aligned}$$

for some constants $C, C_1 > 1$ independent of k, ω.

$\square$

Lemma 5.5. *Let $\mathcal{A}$ be a finitely generated, infinite dimensional, simple unital C^* algebra with a unique tracial state τ. Assume that A has approximation property. Assume that $x_1, \ldots, x_n$ is a family of self-adjoint generators of $\mathcal{A}$. Let H, $\mathcal{M}$ be defined as above. Let m be a positive integer. Let y_m and $P_{r_m}(x_1, \ldots, x_n)$ be chosen as above. Let $R > \max\{\|P_{r_m}(x_1, \ldots, x_n)\|, \|x_1\|, \ldots, \|x_n\|\}$. When r is large enough and ϵ is small enough, for any $\omega > 0$, we have*

$$\nu_2(\Gamma_R^{(top)}(P_{r_m}(x_1, \ldots, x_n) : x_1, \ldots, x_n; k, \epsilon, P_1, \ldots, P_r), \omega) \geq (8C_1\omega)^{-k^2} \cdot \left(\frac{2C}{\omega}\right)^{\frac{-56k^2}{m}}$$

Proof. By Lemma 5.2, when r is large enough and ϵ is small enough, the following hold: $\forall\ k \geq 1$, if

$$(B, A_1, \ldots, A_n) \in \Gamma_R^{(top)}(P_{r_m}(x_1, \ldots, x_n), x_1, \ldots, x_n; k, \epsilon, P_1, \ldots, P_r),$$

then there are some $1 \leq k_1, \ldots, k_m \leq k$ with $\frac{1}{m} - \frac{1}{r} \leq \frac{k_j}{k} \leq \frac{1}{m} + \frac{1}{r}$ for each $1 \leq j \leq m$ and $k_1 + \cdots + k_m = k$, and a unitary matrix U in $\mathcal{U}(k)$ satisfying

$$\|B - U \begin{pmatrix} 1 \cdot I_{k_1} & 0 & \cdots & 0 \\ 0 & 2 \cdot I_{k_2} & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & \cdots & m \cdot I_{k_m} \end{pmatrix} U^*\|_2 \leq \frac{2}{m^3}.$$

Combining with Lemma 5.4, we know that if

$$B \in \Gamma_R^{(top)}(P_{r_m}(x_1, \ldots, x_n) : x_1, \ldots, x_n; k, \epsilon, P_1, \ldots, P_r)$$

then, for any $\omega > 0$,

$$\nu_2(\Omega(B), \omega) \geq (8C_1\omega)^{-k^2} \cdot \left(\frac{2C}{\omega}\right)^{\frac{-56k^2}{m}},$$

where

$$\Omega(B) = \{W^*BW \mid W \in \mathcal{U}(k)\}.$$

Note that $\Omega(B) \subset \Gamma_R^{(top)}(P_{r_m}(x_1, \ldots, x_n) : x_1, \ldots, x_n; k, r, \epsilon)$. It follows that, for any $\omega > 0$,

$$\nu_2(\Gamma_R^{(top)}(P_{r_m}(x_1, \ldots, x_n) : x_1, \ldots, x_n; k, r, \epsilon), \omega) \geq (8C_1\omega)^{-k^2} \cdot \left(\frac{2C}{\omega}\right)^{\frac{-56k^2}{m}}$$

□

Now we have the following result.

Theorem 5.2. *Let $\mathcal{A}$ be a finitely generated, infinite dimensional, simple unital C^* algebra with a unique tracial state τ. Assume that $x_1, \ldots, x_n$ is a family of self-adjoint generators of $\mathcal{A}$. If $\mathcal{A}$ has approximation property, then $\delta_{top}(x_1, \ldots, x_n) \geq 1$.*

Proof. Let H be the Hilbert space $L^2(\mathcal{A},\tau)$. Without loss of generality, we might assume that $\mathcal{A}$ is faithfully represented on the Hilbert space H. Let $\mathcal{M}$ be the von Neumann algebra generated by $\mathcal{A}$ on H. It is not hard to see that $\mathcal{M}$ is a diffuse von Neumann algebra with a tracial state τ. For each positive integer m, there is a family of mutually orthogonal projections $p_1,\ldots,p_m$ in $\mathcal{M}$ such that $\tau(p_j)=1/m$ for $1\le j\le m$. Let

$$y_m = 1\cdot p_1 + 2\cdot p_2 + \cdots + m\cdot p_m = \sum_{j=1}^{m} j\cdot p_j.$$

Let $\{P_r(x_1,\ldots,x_n)\}_{r=1}^{\infty}$ be defined as in section 2.3. Thus $\{P_r(x_1,\ldots,x_n)\}_{r=1}^{\infty}$ is dense in $\mathcal{M}$ with respect to the strong operator topology. Hence, for each $m\ge 1$, there is some self-adjoint element $P_{r_m}(x_1,\ldots,x_n)$ in $\mathcal{A}$ such that

$$\|y_m - P_{r_m}(x_1,\ldots,x_n)\|_2 \le \frac{1}{m^3}.$$

By Lemma 5.5, for any $\omega>0$, when r is large enough and ϵ is small enough, we have for some constants $C_1, C>1$ independent of k,ω

$$\nu_2(\Gamma_R^{(top)}(P_{r_m}(x_1,\ldots,x_n):x_1,\ldots,x_n;k,\epsilon,P_1,\ldots,P_r),\omega) \ge (8C_1\omega)^{-k^2}\cdot\left(\frac{2C}{\omega}\right)^{\frac{-56k^2}{m}}$$

Therefore,

$$\tilde{\delta}_{top}(P_{r_m}(x_1,\ldots,x_n):x_1,\ldots,x_n) \ge 1-\frac{56}{m}.$$

By Proposition 5.1, we get

$$\delta_{top}(P_{r_m}(x_1,\ldots,x_n):x_1,\ldots,x_n) \ge 1-\frac{56}{m}.$$

By Lemma 5.1,

$$\delta_{top}(x_1,\ldots,x_n) \ge 1-\frac{56}{m}.$$

Since m is an arbitrary positive integer, we obtain

$$\delta_{top}(x_1,\ldots,x_n)\ge 1.$$

□

5.4. Values of topological free entropy dimensions in some unital C* algebras. In this subsection, we are going to compute the values of topological free entropy dimensions in some unital C* algebras by using the results from preceding subsection.

Theorem 5.3. *Let $\mathcal{A}_\theta$ be an irrational rotation C* algebra. Then*

$$\delta_{top}(x_1,\ldots,x_n)=1$$

where $x_1,\ldots,x_n$ is a family of self-adjoint operators that generates $\mathcal{A}_\theta$.

Proof. Note that $\mathcal{A}_\theta$ is an infinite dimensional, unital simple C* algebra with a unique tracial state τ. By [**24**] or [**11**] and Theorem 5.1, we know that

$$\delta_{top}(x_1, \ldots, x_n) \le 1.$$

It follows from [**18**] that $\mathcal{A}_\theta$ has approximation property. Therefore

$$\delta_{top}(x_1, \ldots, x_n) \ge 1.$$

Hence

$$\delta_{top}(x_1, \ldots, x_n) = 1.$$

□

Theorem 5.4. *Let $\mathcal{A}$ be a UHF algebra (uniformly hyperfinite C* algebra). Then*

$$\delta_{top}(x_1, \ldots, x_n) = 1$$

where $x_1, \ldots, x_n$ is a family of self-adjoint operators that generates $\mathcal{A}$.

Proof. By [**17**], we know that $\mathcal{A}$ is generated by two self-adjoint elements. It is not hard to see that $\mathcal{A}$ is an infinite dimensional, unital simple C* algebra with a unique tracial state τ. By [**24**] or [**11**] and Theorem 5.1, we know that

$$\delta_{top}(x_1, \ldots, x_n) \le 1.$$

It is easy to check that $\mathcal{A}$ has approximation property. Therefore

$$\delta_{top}(x_1, \ldots, x_n) \ge 1.$$

Hence

$$\delta_{top}(x_1, \ldots, x_n) = 1.$$

□

Recall that for any sequence $(\mathcal{A}_m)_{m=1}^\infty$ of C* algebras,we can introduce two C* algebras

$$\prod_m \mathcal{A}_m = \{(a_m)_{m=1}^\infty \mid a_m \in \mathcal{A}_m, \ \sup_{m\in\mathbb{N}} \|a_m\| < \infty\}$$
$$\sum_m \mathcal{A}_m = \{(a_m)_{m=1}^\infty \mid a_m \in \mathcal{A}_m, \ \lim_{m\to\mathbb{N}} \|a_m\| = 0\}$$

The norm in the quotient C* algebra $\prod_m \mathcal{A}_m / \sum_m \mathcal{A}_m$ is given by

$$\|\rho((a_m)_{m=1}^\infty)\| = \limsup_{m\to\infty} \|x_m\|,$$

where ρ is the quotient map from $\prod_m \mathcal{A}_m$ onto $\sum_m \mathcal{A}_m$.

If $\mathcal{A}$ is an exact C* algebra, then the sequence

$$0 \to \mathcal{A} \otimes_{min} \sum_m M_m(\mathbb{C}) \to \mathcal{A} \otimes_{min} \prod_m M_m(\mathbb{C}) \to \mathcal{A} \otimes_{min} (\prod_m M_m(\mathbb{C}) / \sum_m M_m(\mathbb{C})) \to 0$$

is exact. Therefore, we have the following natural identification

$$\mathcal{A} \otimes_{min} (\prod_m M_m(\mathbb{C}) / \sum_m M_m(\mathbb{C})) = (\mathcal{A} \otimes_{min} \prod_m M_m(\mathbb{C}))/(\mathcal{A} \otimes_{min} \sum_m M_m(\mathbb{C})).$$

On the other hand, we have the following natural embedding

$$\mathcal{A} \otimes_{min} \prod_m M_m(\mathbb{C}) \subseteq \prod_m \mathcal{M}_m(\mathcal{A})$$

and the identification

$$\mathcal{A} \otimes_{min} \sum_m M_m(\mathbb{C}) = \sum_m \mathcal{M}_m(\mathcal{A})$$

Thus we have for any exact C* algebra $\mathcal{A}$ a natural embedding

$$\psi : \quad \mathcal{A} \otimes_{min} (\prod_m M_m(\mathbb{C}) / \sum_m M_m(\mathbb{C}) \subseteq \prod_m M_m(\mathcal{A}) / \sum_m M_m(\mathcal{A}).$$

LEMMA 5.6. *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are unital C* algebras and ρ is an unital embedding*

$$\rho : \quad \mathcal{A} \to \prod_m M_m(\mathcal{B}) / \sum_m M_m(\mathcal{B}).$$

Suppose that $x_1, \ldots, x_n$ is a family of elements in $\mathcal{A}$. Suppose r is a positive integer and $\{P_j(x_1, \ldots, x_n)\}_{j=1}^r$ is a family of noncommutative polynomials of $x_1, \ldots, x_n$. Then there are some $k \in \mathbb{N}$ and $a_1^{(k)}, \ldots, a_n^{(k)}$ in $\mathcal{M}_k(\mathcal{B})$ so that

$$|\|P_j(a_1^{(k)}, \ldots, a_n^{(k)})\| - \|P_j(x_1, \ldots, x_n)\|| \le \frac{1}{r}, \quad \forall\ 1 \le j \le r.$$

PROOF. We might assume that

$$\rho(x_i) = [(x_i^{(m)})_m] \ \in \ \prod_m M_m(\mathcal{B}) / \sum_m M_m(\mathcal{B}), \quad \forall\ 1 \le i \le n.$$

By the definition of $\prod_m M_m(\mathcal{B}) / \sum_m M_m(\mathcal{B})$, there are some positive integers $m_1 \le m_2$ such that

$$|(\sup_{m_1 \le l \le m_2} \|P_j(x_1^{(l)}, \ldots, x_n^{(l)})\|) - \|P_j(x_1, \ldots, x_n)\|| \le \frac{1}{r}, \quad \forall\ 1 \le j \le r.$$

Let $k = \sum_{j=m_1}^{m_2} j$ and

$$a_i^{(k)} = \oplus_{l=m_1}^{m_2} x_i^{(l)} \in \mathcal{M}_k(\mathcal{B}), \qquad \forall\ 1 \le i \le n.$$

Then, it is not hard to check that

$$|\|P_j(a_1^{(k)}, \ldots, a_n^{(k)})\| - \|P_j(x_1, \ldots, x_n)\|| \le \frac{1}{r}, \quad \forall\ 1 \le j \le r.$$

$\square$

THEOREM 5.5. *Let $p \geq 2$ be a positive integer and F_p be the free group on p generators. Let $C^*_{red}(F_p) \otimes_{min} C^*_{red}(F_p)$ be a minimal tensor product of two reduced C^* algebras of free groups F_p. Then*

$$\delta_{top}(x_1, \ldots, x_n) = 1,$$

where $x_1, \ldots, x_n$ is any family of self-adjoint generators of $C^*_{red}(F_p) \otimes_{min} C^*_{red}(F_p)$.

PROOF. Note that $C^*_{red}(F_p) \otimes_{min} C^*_{red}(F_p)$ is an infinite dimensional, unital simple C* algebra with a unique tracial state. By the result from [**5**] or [**11**] and Theorem 5.1, Theorem 5.2, to show $\delta_{top}(x_1, \ldots, x_n) = 1$, we need only to show that $C^*_{red}(F_p) \otimes_{min} C^*_{red}(F_p)$ has approximation property. Therefore, it suffices to show the following: Let $R > \max\{\|x_1\|, \ldots, \|x_n\|\}$. For any $r \geq 1$, there is some $k \in \mathbb{N}$ so that

$$\Gamma_R^{(top)}(x_1, \ldots, x_n; k, \frac{1}{r}, P_1, \ldots, P_r) \neq \emptyset.$$

By the result from [**9**], we know there is a unital embedding

$$\phi_1: \quad C^*_{red}(F_p) \rightarrow \prod_m \mathcal{M}_m(\mathbb{C}) / \sum_m \mathcal{M}_m(\mathbb{C}),$$

which induce a unital embedding

$$\phi_2: \quad C^*_{red}(F_p) \otimes_{min} C^*_{red}(F_p) \rightarrow C^*_{red}(F_p) \otimes_{min} (\prod_m \mathcal{M}_m(\mathbb{C}) / \sum_m \mathcal{M}_m(\mathbb{C}))$$

Note that $C^*_{red}(F_p)$ is an exact C* algebra. From the explanation preceding the theorem it follows that there is a unital embedding

$$\phi_3: \quad C^*_{red}(F_p) \otimes_{min} C^*_{red}(F_p) \rightarrow \prod_m \mathcal{M}_m(C^*_{red}(F_p)) / \sum_m \mathcal{M}_m(C^*_{red}(F_p)).$$

By Lemma 5.6, for a family of elements $x_1, \ldots, x_n$ in $C^*_{red}(F_p) \otimes_{min} C^*_{red}(F_p)$ and $r \geq 1$, there are some $m \in \mathbb{N}$ and some $a_1^{(m)}, \ldots, a_n^{(m)}$ in $\mathcal{M}_m(C^*_{red}(F_p))$ so that $\max\{\|a_1\|, \ldots, \|a_n\|\} < R$ and

$$|\|P_j(a_1^{(m)}, \ldots, a_n^{(m)})\| - \|P_j(x_1, \ldots, x_n)\|| \leq \frac{1}{2r}, \qquad \forall\ 0 \leq j \leq r.$$

On the other hand, by the existence of embedding

$$\phi_1: \quad C^*_{red}(F_p) \rightarrow \prod_{m'} \mathcal{M}_{m'}(\mathbb{C}) / \sum_{m'} \mathcal{M}_{m'}(\mathbb{C}),$$

it follows that there is a unital embedding

$$\phi_4: \ \mathcal{M}_m(C^*_{red}(F_p)) = \mathcal{M}_m(\mathbb{C}) \otimes_{min} C^*_{red}(F_p) \rightarrow \mathcal{M}_m(\mathbb{C}) \otimes_{min} (\prod_{m'} \mathcal{M}_{m'}(\mathbb{C}) / \sum_{m'} \mathcal{M}_{m'}(\mathbb{C}))$$

But

$$\mathcal{M}_m(\mathbb{C}) \otimes_{min} (\prod_{m'} \mathcal{M}_{m'}(\mathbb{C}) / \sum_{m'} \mathcal{M}_{m'}(\mathbb{C})) = \prod_{m'} \mathcal{M}_{m'm}(\mathbb{C}) / \sum_{m'} \mathcal{M}_{m'm}(\mathbb{C}).$$

Hence for such $a_1^{(m)}, \ldots, a_n^{(m)}$ in $\mathcal{M}_m(C^*_{red}(F_p))$ and $r \geq 1$, by Lemma 5.6, there are some $k \in \mathbb{N}$ and $A_1, \ldots, A_n$ in $\mathcal{M}_k(\mathbb{C})$ so that $\max\{\|A_1\|, \ldots, \|A_n\|\} < R$ and

$$|\|P_j(a_1^{(m)}, \ldots, a_n^{(m)})\| - \|P_j(A_1, \ldots, A_n)\|| \leq \frac{1}{2r}, \qquad \forall\ 0 \leq j \leq r.$$

Altogether, we have

$$|\|P_j(x_1, \ldots, x_n)\| - \|P_j(A_1, \ldots, A_n)\|| \leq \frac{1}{r}, \qquad \forall\ 0 \leq j \leq r,$$

which implies that $C^*_{red}(F_p) \otimes_{min} C^*_{red}(F_p)$ has approximation property.

Hence

$$\delta_{top}(x_1, \ldots, x_n) = 1,$$

for any family of self-adjoint elements $x_1, \ldots, x_n$ that generates $C^*_{red}(F_p) \otimes_{min} C^*_{red}(F_p)$. $\square$

Theorem 5.6. *Suppose that $\mathcal{K}$ be the C^* algebra consisting of all compact operators on a separable Hilbert space H. Suppose $\mathcal{A} = \mathbb{C} \bigoplus \mathcal{K}$ is the unitization of $\mathcal{K}$. If $x_1, \ldots, x_n$ is a family of self-adjoint elements that generate $\mathcal{A}$ as a C^* algebra, then*

$$\delta_{top}(x_1, \ldots, x_n) = 0.$$

Proof. By [**17**], we know that unital C* algebra $\mathcal{A}$ is generated by two self-adjoint elements in $\mathcal{A}$. Note that $\mathcal{A}$ has a unique trace τ, which is defined by

$$\tau((\lambda, x)) = \lambda, \qquad \forall\ (\lambda, x) \in \mathcal{A}.$$

By Theorem 5.1, it is not hard to see that

$$\delta_{top}(x_1, \ldots, x_n) = 0,$$

where $x_1, \ldots, x_n$ is a family of self-adjoint generators of $\mathcal{A}$.

$\square$

6. Topological free orbit dimension of C* algebras

Assume that $\mathcal{A}$ is a unital C*-algebra. Let $x_1, \ldots, x_n$, $y_1, \ldots, y_m$ be self-adjoint elements in $\mathcal{A}$. Let $\mathbb{C}\langle X_1, \ldots, X_n, Y_1, \ldots, Y_m\rangle$ be the noncommutative polynomials in the indeterminates $X_1, \ldots, X_n$, $Y_1, \ldots, Y_m$. Let $\{P_r\}_{r=1}^{\infty}$ be the collection of all noncommutative polynomials in $\mathbb{C}\langle X_1, \ldots, X_n, Y_1, \ldots, Y_m\rangle$ with rational coefficients.

6.1. Unitary orbits of balls in $\mathcal{M}_k(\mathbb{C})^n$. We let $\mathcal{M}_k(\mathbb{C})$ be the $k \times k$ full matrix algebra with entries in $\mathbb{C}$, and $\mathcal{U}(k)$ be the group of all unitary matrices in $\mathcal{M}_k(\mathbb{C})$. Let $\mathcal{M}_k(\mathbb{C})^n$ denote the direct sum of n copies of $\mathcal{M}_k(\mathbb{C})$. Let $\mathcal{M}_k^{s.a}(\mathbb{C})$ be the subalgebra of $\mathcal{M}_k(\mathbb{C})$ consisting of all self-adjoint matrices of $\mathcal{M}_k(\mathbb{C})$. Let $(\mathcal{M}_k^{s.a}(\mathbb{C}))^n$ be the direct sum of n copies of $\mathcal{M}_k^{s.a}(\mathbb{C})$.

For every $\omega > 0$, we define the ω-orbit-$\|\cdot\|$-ball $\mathcal{U}(B_1, \ldots, B_n; \omega)$ centered at $(B_1, \ldots, B_n)$ in $\mathcal{M}_k(\mathbb{C})^n$ to be the subset of $\mathcal{M}_k(\mathbb{C})^n$ consisting of all $(A_1, \ldots, A_n)$ in $\mathcal{M}_k(\mathbb{C})^n$ such that there exists some unitary matrix W in $\mathcal{U}(k)$ satisfying

$$\|(A_1, \ldots, A_n) - (WB_1W^*, \ldots, WB_nW^*)\| < \omega.$$

6.2. Norm-microstate space. For all integers $r, k \geq 1$, real numbers $R, \epsilon > 0$ and non-commutative polynomials $P_1, \ldots, P_r$, we let

$$\Gamma_R^{(top)}(x_1, \ldots, x_n : y_1, \ldots, y_m; k, \epsilon, P_1, \ldots, P_r)$$

be as defined as in section 2.4.

6.3. Topological free orbit dimension.

DEFINITION 6.1. *For $\omega > 0$, we define the covering number*

$$o_\infty(\Gamma_R^{(top)}(x_1, \ldots, x_n : y_1, \ldots, y_p; k, \epsilon, P_1, \ldots, P_r), \omega)$$

to be the minimal number of ω-orbit–$\|\cdot\|$-balls that cover $\Gamma_R^{(top)}(x_1, \ldots, x_n : y_1, \ldots, y_p; k, \epsilon, P_1, \ldots, P_r)$ with the centers of these ω-orbit-$\|\cdot\|$-balls in $\Gamma_R^{(top)}(x_1, \ldots, x_n : y_1, \ldots, y_p; k, \epsilon, P_1, \ldots, P_r)$

For each function $f: \mathbb{N} \times \mathbb{N} \times \mathbb{R}^+ \to \mathbb{R}$, we define,

$$\begin{aligned} &\mathfrak{k}_f(x_1, \ldots, x_n : y_1, \ldots, y_p; \omega, R) \\ &\quad = \inf_{r \in \mathbb{N}, \epsilon > 0} \limsup_{k \to \infty} f(o_\infty(\Gamma_R^{(top)}(x_1, \ldots, x_n : y_1, \ldots, y_p; k, \epsilon, P_1, \ldots, P_r), \omega), k, \omega) \end{aligned}$$

and

$$\mathfrak{k}_f(x_1, \ldots, x_n : y_1, \ldots, y_p; \omega) = \sup_{R>0} \mathfrak{k}_f(x_1, \ldots, x_n : y_1, \ldots, y_p; \omega, R)$$

$$\mathfrak{k}_f(x_1, \ldots, x_n : y_1, \ldots, y_p) = \limsup_{\omega \to 0^+} \mathfrak{k}_f(x_1, \ldots, x_n : y_1, \ldots, y_p; \omega),$$

where $\mathfrak{k}_f(x_1, , \ldots, x_n : y_1, \ldots, y_p)$ is called the topological $f(\cdot)$-free-orbit-dimension *of $x_1, \ldots, x_n$ in the presence of $y_1, \ldots, y_p$.*

6.4. Topological free entropy dimension and topological free orbit dimension. The following result follows directly from the definitions of topological free entropy dimension and topological free orbit dimension of n-tuple of self-adjoint elements in a C* algebra.

THEOREM 6.1. *Suppose that $\mathcal{A}$ is a unital C^* algebra and $x_1, \ldots, x_n$ is a family of self-adjoint elements of $\mathcal{A}$. Let $f : \mathbb{N} \times \mathbb{N} \times \mathbb{R}^+ \to \mathbb{R}$ be defined by*

$$f(s, k, \omega) = \frac{\log s}{-k^2 \log \omega}$$

for $s, k \in \mathbb{N}$, $\omega > 0$. Then

$$\delta_{top}(x_1, \ldots, x_n) \leq \mathfrak{k}_f(x_1, \ldots, x_n) + 1.$$

7. Topological free orbit dimension of one variable

We recall the packing number of a set in a metric space as follows.

DEFINITION 7.1. *Suppose that X is a metric space with a metric distance d. (i) The packing number of a set K by ω-balls in X, denoted by $P(K, \omega)$, is the maximal cardinality of the subsets*

F in K satisfying for all a,b in F either $a=b$ or $d(a,b)\geq\omega$. (ii) The packing dimension of the set K in X, denoted by $d(K)$, is defined by

$$d(K)=\limsup_{\omega\to 0^+}\frac{\log(P(K,\omega))}{-\log\omega}.$$

7.1. Upper-bound of the topological free orbit dimension of one variable. Suppose that $x=x^*$ is a self-adjoint element in a unital C* algebra $\mathcal{A}$ and $\sigma(x)$ is the spectrum of x in $\mathcal{A}$.

For any $\omega>0$, let $m=P(K,\omega)$ be the packing number of $\sigma(x)$ in $\mathbb{R}$. Thus there exists a family of elements $\lambda_1,\ldots,\lambda_m$ in $\sigma(x)$ such that (i) $|\lambda_i-\lambda_j|\geq\omega$ for all $1\leq i\neq j\leq m$; and (ii) for any λ in $\sigma(x)$, there is some λ_j with $1\leq j\leq m$ satisfying $|\lambda-\lambda_j|\leq\omega$.

Lemma 7.1. *For any given $R>\|x\|$, when r is large enough and ϵ is small enough, we have*

$$\limsup_{k\to\infty}\frac{\log o_\infty(\Gamma_R^{(top)}(x;k,\epsilon,P_1,\ldots,P_r),3\omega)}{\log k}\leq m.$$

Proof. By Theorem 3.1, there exist some $r_0\geq 1$ and $\epsilon_0>0$ such that the following holds: when $r>r_0$, $\epsilon<\epsilon_0$, for any A in $\Gamma_R^{(top)}(x;k,\epsilon,P_1,\ldots,P_r)$, there are positive integers $1\leq k_1,\ldots,k_m\leq k$ with $k_1+\cdots+k_m=k$ and some unitary matrix U in $\mathcal{M}_k(\mathbb{C})$ satisfying

$$\|U^*AU-\begin{pmatrix}\lambda_1 I_{k_1} & 0 & \cdots & 0\\ 0 & \lambda_2 I_{k_2} & \cdots & 0\\ \cdots & \cdots & \ddots & \cdots\\ 0 & 0 & \cdots & \lambda_m I_{k_m}\end{pmatrix}\|\leq 2\omega,$$

where I_{k_j} is the $k_j\times k_j$ identity matrix for $1\leq j\leq m$.

Let

$$\Omega(k_1,\ldots,k_m)=\left\{U^*\begin{pmatrix}\lambda_1 I_{k_1} & 0 & \cdots & 0\\ 0 & \lambda_2 I_{k_2} & \cdots & 0\\ \cdots & \cdots & \ddots & \cdots\\ 0 & 0 & \cdots & \lambda_m I_{k_m}\end{pmatrix}U \mid U \text{ is in } \mathcal{U}_k\right\}.$$

Let $\mathcal{J}$ be the set consisting of all these $(k_1,\ldots,k_m)\in\mathbb{N}^m$ with $k_1+\cdots+k_m=k$. Then the cardinality of the set $\mathcal{J}$ is equal to

$$\frac{(k-1)!}{(m-1)!(k-m)!}.$$

Then

$$\Gamma_R^{(top)}(x;k,\epsilon,P_1,\ldots,P_r)$$

is contained in 2ω-neighborhood of the set

$$\bigcup_{(k_1,\ldots,k_m)\in\mathcal{J}}\Omega(k_1,\ldots,k_m).$$

It follows that

$$o_\infty(\Gamma_R^{(top)}(x;k,\epsilon,P_1,\ldots,P_r),3\omega) \le o_\infty(\bigcup_{(k_1,\ldots,k_m)\in\mathcal{J}} \Omega(k_1,\ldots,k_m),\omega) \le |\mathcal{J}| = \frac{(k-1)!}{(m-1)!(k-m)!}.$$

Therefore,

$$\begin{aligned}\limsup_{k\to\infty} \frac{\log o_\infty(\Gamma_R^{(top)}(x;k,\epsilon,P_1,\ldots,P_r),3\omega)}{\log k} &= \limsup_{k\to\infty} \frac{\log o_\infty(\Gamma_R^{(top)}(x;k,\epsilon,P_1,\ldots,P_r),\omega)}{\log k} \\ &\le \limsup_{k\to\infty} \frac{\log\left(\frac{(k-1)!}{(m-1)!(k-m)!}\right)}{\log k} = m-1.\end{aligned}$$

□

7.2. Lower-bound. Suppose that $x = x^*$ is a self-adjoint element in a unital C* algebra $\mathcal{A}$ and $\sigma(x)$ is the spectrum of x in $\mathcal{A}$.

LEMMA 7.2. *We have*

$$\limsup_{k\to\infty} \frac{\log o_\infty(\Gamma_R^{(top)}(x;k,\epsilon,P_1,\ldots,P_r),\frac{\omega}{3})}{\log k} \ge m-1.$$

PROOF. For any $\omega > 0$, let $m = P(K,\omega)$ be the packing number of $\sigma(x)$ in $\mathbb{R}$. Thus there exists a family of elements $\lambda_1,\ldots,\lambda_m$ in $\sigma(x)$ such that (i) $|\lambda_i - \lambda_j| \ge \omega$ for all $1 \le i \ne j \le m$; and (ii) for any λ in $\sigma(x)$, there is some λ_j with $1 \le j \le m$ satisfying $|\lambda - \lambda_j| \le \omega$.

For any $R > \|x\|$, $r \ge 1$ and $\epsilon > 0$, by functional calculus, there are $\lambda_{m+1},\ldots,\lambda_n$ in $\sigma(x)$ such that for every $1 \le t_1,\ldots,t_m \le k-n$ with $2nt_1 + \ldots + 2nt_m = k-n$, the matrix

$$A = diag(\lambda_1 I_{2nt_1},\lambda_2 I_{2nt_2},\ldots,\lambda_m I_{2nt_m},\lambda_1,\ldots,\lambda_m,\ldots,\lambda_n)$$
$$\text{is in } \Gamma_R^{(top)}(x;k,\epsilon,P_1,\ldots,P_r), \tag{7.2.1}$$

where we assume that $2n|(k-n)$.

Let $\mathcal{J}$ be the set consisting of all these $(t_1,\ldots,t_m) \in \mathbb{N}^m$ with $2nt_1 + \ldots + 2nt_m = k-n$. Then the cardinality of the set $\mathcal{J}$ is equal to

$$\frac{\left(\frac{k-n}{2n}-1\right)!}{\left(\frac{k-n}{2n}-m\right)!(m-1)!}.$$

By Weyl's theorem in [**27**] on the distance of unitary orbits of two self-adjoint matrices, for any two distinct elements

$$(s_1,\ldots,s_m) \qquad \text{and} \qquad (t_1,\ldots,t_m)$$

in $\mathcal{J}$ and any W in $\mathcal{U}(k)$, we have

$$\|A_1 - WA_2W^*\| \ge \omega,$$

where

$$A_1 = diag(\lambda_1 I_{2nt_1}, \lambda_2 I_{2nt_2}, \ldots, \lambda_m I_{2nt_m}, \lambda_1, \ldots, \lambda_m, \ldots, \lambda_n)$$
$$A_2 = diag(\lambda_1 I_{2ns_1}, \lambda_2 I_{2ns_2}, \ldots, \lambda_m I_{2ns_m}, \lambda_1, \ldots, \lambda_m, \ldots, \lambda_n)$$

are two diagonal self-adjoint matrices in $\mathcal{M}_k(\mathbb{C})$. Combining with (7.2.1), we have

$$o_\infty(\Gamma_R^{(top)}(x; k, \epsilon, P_1, \ldots, P_r), \frac{\omega}{3}) \ge |\mathcal{J}| \ge \frac{\left(\frac{k-n}{2n} - 1\right)!}{\left(\frac{k-n}{2n} - m\right)!(m-1)!}.$$

Hence

$$\limsup_{k\to\infty} \frac{\log o_\infty(\Gamma_R^{(top)}(x; k, \epsilon, P_1, \ldots, P_r), \frac{\omega}{3})}{\log k} \ge \limsup_{k\to\infty} \frac{\log \frac{\left(\frac{k-n}{2n}-1\right)!}{\left(\frac{k-n}{2n}-m\right)!(m-1)!}}{\log k} = m - 1.$$

□

7.3. Topological free orbit dimension of one self-adjoint element.

Theorem 7.1. *Suppose that $x = x^*$ is a self-adjoint element in a unital C^* algebra $\mathcal{A}$ and $\sigma(x)$ is the spectrum of x in $\mathcal{A}$. Let $d(\sigma(x))$ be the packing dimension of the set $\sigma(x)$ in $\mathbb{R}$. Let $f : \mathbb{N} \times \mathbb{N} \times \mathbb{R}^+ \to \mathbb{R}$ be defined by*

$$f(s, k, \omega) = \frac{\log\left(\frac{\log s}{\log k}\right)}{-\log \omega}$$

for $s, k \in \mathbb{N}$, $\omega > 0$. Then

$$\mathfrak{k}_f(x) = d(\sigma(x)).$$

Proof. The result follows directly from Lemma 7.1, Lemma 7.2 and Definition 7.1. □

Theorem 7.2. *Suppose that $x = x^*$ is a self-adjoint element in a unital C^* algebra $\mathcal{A}$. Let $f : \mathbb{N} \times \mathbb{N} \times \mathbb{R}^+ \to \mathbb{R}$ be defined by*

$$f(s, k, \omega) = \frac{\log s}{-k^2 \log \omega}$$

for $s, k \in \mathbb{N}$, $\omega > 0$. Then

$$\mathfrak{k}_f(x) = 0.$$

Proof. The result follows directly from Lemma 7.1 and Definition 7.1. □

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